# Aspects of Domination Theory

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# Preface

This thesis is the result of my PhD-study at the Department of Mathematical Sciences, Aalborg University, Denmark. The topic of the thesis is domination theory.

The thesis contains the following 9 papers :

- Frendrup, A., Henning, M. A., Randerath, B. & Vestergaard, P. D. A New Upper Bound on the Domination Number of a Graph. *Discrete Mathematics.* Accepted for publication.
- Frendrup, A., Henning, M. A., Randerath, B. & Vestergaard, P. D. On a Conjecture about Inverse Domination in Graphs. *Ars Combinatoria.*. Accepted for publication.
- Frendrup, A., Pedersen, A. S., Sapozhenko, A. A. & Vestergaard, P. D. Merrifield-Simmons Index and Minimum Number of Independent Sets in Short Trees. *Ars Combinatoria.* Submitted.
- Frendrup, A. The Number of Independent Sets in Graphs with Minimum Degree 2 or 3 or Graphs with $m$ Edges.
- Finbow, A., Frendrup, A. & Vestergaard, P. D. Total Well Dominated Trees.
- Frendrup, A., Henning, M. A., & Vestergaard, P. D. Total Domination in Partitioned Trees and Partitioned Graphs with Minimum Degree Two. *Journal of Global Optimization.* Accepted for publication.
- Frendrup, A., Vestergaard, P. D. & Yeo, A. Total Domination in Partitioned Graphs. *Discrete Mathematics.* Submitted.
- Frendrup, A., Tuza, Z. & Vestergaard, P. D. Distance Domination in Partitioned Graphs.
- Frendrup, A. Total Domination (Without Isolated Vertices) in Partitioned Graphs.

The published papers have been included only changing layout and a few minor details. In each paper the necessary notation is presented. Thus each chapter can be read independently of each other but some parts may be repeated in different papers.

## Acknowledgements

I am grateful to my supervisor Preben Dahl Vestergaard. Thank you for proposing many of the problems considered in this thesis, helpful discussions and inspiration, for doing cooperative work with me and for making me part of great research environments.

I wish to thank Art Finbow, Michael A. Henning, Anders Sune Pedersen, Bert Randerath, Alexander Sapozhenko, Zsolt Tuza and Anders Yeo for great collaboration, helpful discussions and for introducing new problems to me.

Aalborg, March 2008 Allan Frendrup

## Summary in Danish

Denne afhandling, som består af 9 artikler, omhandler forskellige aspekter af dominansteori. Dominansteori er et af de emner indenfor grafteori, hvor der er sket stor udvikling gennem de sidste årtier.

Nogle af de mest klassiske resultater indenfor dominansteori er øvre grænser for dominanstallet for grafer med en given minimumsvalens. I artiklen "Paths, Stars and the Number Three" blev det vist, at grafer med minimumsvalens på mindst tre har dominanstal, som højst er tre ottendedele af dens orden. I artiklen "A New Upper Bound on the Domination Number of a Graph" udvides denne grænse til en øvre grænse for grafer med minimumsvalens på mindst to. Den nye grænse er tre ottendedele af grafens orden plus parametre afhængende af placeringen af valens to punkterne i grafen.

I artiklen "Inverse Domination in Graphs" viste V. R. Kulli og S. C. Sigarkant, at det inverse dominanstal ikke er større end uafhængighedstallet for en graf. Det viste sig dog, at der var en uoprettelig fejl i deres bevis, og siden har G. S. Domke, J. E. Dunbar og L. R. Markus stillet problemet som en formodning i artiklen "The Inverse Domination Number of a Graph". Ideerne fra det fejlagtige bevis kan dog benyttes til at bevise formodningen for visse familier af grafer. Dette er gjort i artiklen "On a Conjecture about Inverse Domination in Graphs", og yderligere er der brugt nye ideer til at bevise formodningen for forskellige familier af grafer.

Antallet af uafhængige mængder i grafer er et emne, som både er blevet betragtet indenfor kemi og grafteori. Kemikere har den formodning, at antallet af uafhængige mængder i et stofs molekylegraf kan beskrive egenskaber for stoffet. Afhandlingen indeholder to artikler, "Merrifield-Simmons Index and Independent Sets in Short Trees" og "The Number of Independent Sets in Graphs with Minimum Degree 2 or 3 or Graphs with $m$ Edges", som beskæftiger sig med antallet af uafhængige mængder i grafer.

Inden for dominansteori har der været en del interesse for at karakterisere grafer, hvori enhver punktmængde eller kantmængde, som opfylder bestemte betingelser, alle har samme kardinalitet. Blandt andet har man karakteriseret grafer, hvori alle maksimale uafhængige mængder, maksimale parringer og minimale dominerende mængder har samme kardinalitet. I artiklen "Total well dominated trees" karakteriseres de træer, hvori alle minimal total dominerende mængder har samme kardinalitet.

I artiklen "Partitions and Domination in Graphs" introducerede B. Hartnell og P. D. Vestergaard konceptet opdelt dominans. Siden da har flere, heriblandt

M. A. Henning, S. Seager og Z. Tuza, betragtet samme emne for forskellige former for dominans. Ved opdelt domians betragtes enhver tænkelig opdeling i et givent antal klasser af punktmænden for en graf, og der ses på summen af dominanstal for hver klasse. I afhandlingen betragtes opdelt dominans for afstandsdominans og total dominans i de fire artikler "Total Domination in Partitioned Trees and Partitioned Graphs With Minimum Degree Two", "Total Domination in Partitioned Graphs", "Distance Domination in Partitioned Graphs" og "Total Domination (Without Isolated Vertices) in Partitioned Graphs".

## Summary in English

This thesis consists of 9 articles treating aspects of domination theory. Domination theory is one of the fields within graph theory where there has been much research within the last decades.

Some of the classic results within domination theory are upper bounds on the domination number for graphs with a given minimum valency. In the article "Paths, Stars and the Number Three" it was proven that graphs of minimum valency three have domination number at most three eights the order. In the article "A New Upper Bound on the Domination Number of a Graph" this result is extended to graphs with minimum valency two. The new bound is three eights the order plus a parameter depending on how the vertices of degree two are located in the graph.

In the article "Inverse Domination in Graphs" V. R. Kulli and S. C. Sigarkant showed that the inverse domination number of a graph is at most equal to the independence number of the graph. However the proof is incorrect and later G. S. Domke, J. E. Dunbar, and L. R. Markus formally stated this "result" as a conjecture in the article "The Inverse Domination Number of a Graph". Some of the ideas from the incorrect proof can be used to prove the conjecture for some families of graphs. This is done in "On a Conjecture about Inverse Domination in Graphs" and further more some new ideas are applied to prove the conjecture for different kind of graph families.

The number of independent sets in a graph is a subject that has been considered both within chemistry and graph theory. Some chemists presume that the number of independent sets in a molecular graph indicates properties of the chemical compound. This thesis contains two articles, "Merrifield-Simmons Index and Independent Sets in Short Trees" and "The Number of Independent Sets in Graphs with Minimum Degree 2 or 3 or Graphs with $m$ Edges", where the number of independent sets in graphs is considered.

In domination theory there has been some interest in characterising graphs where every vertex set or edge set with a given property all have the same cardinality. For instance there has been some research to characterise graphs where all maximal independent sets, maximal matchings, and minimal dominating sets have the same cardinality. In the article "Total well dominated trees" a characterisation is given for trees where all minimal total dominating sets have the same cardinality.

In the article "Partitions and Domination in Graphs" B. Hartnell and P. D. Vestergaard introduced the concept of partitioned domination. Since then other

researchers, including M. A. Henning, S. Seager, and Z. Tuza, have studied this subject for different kinds of domination. In the thesis partitioned domination is considered for distance domination and total domination in the four articles: "Total Domination in Partitioned Trees and Partitioned Graphs with Minimum Degree Two", "Total Domination in Partitioned Graphs", "Distance Domination in Partitioned Graphs", and "Total Domination (Without Isolated Vertices) in Partitioned Graphs".

# A New Upper bound on the domination number of a Graph

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**Abstract**

A set $S$ of vertices in a graph $G$ is a dominating set of $G$ if every vertex of $V(G) \setminus S$ is adjacent to some vertex in $S$. The minimum cardinality of a dominating set of $G$ is the domination number of $G$, denoted $\gamma(G)$. Let $P_n$ and $C_n$ denote a path and a cycle, respectively, on $n$ vertices. Let $k_1(F)$ and $k_2(F)$ denote the number of components of a graph $F$ that are isomorphic to a graph in the family $\{P_3, P_4, P_5, C_5\}$ and $\{P_1, P_2\}$, respectively. Let $\mathcal{L}$ be the set of vertices of $G$ of degree more than 2, and let $G - \mathcal{L}$ be the graph obtained from $G$ by deleting the vertices in $\mathcal{L}$ and all edges incident with $\mathcal{L}$. McCuaig and Shepherd [5] showed that if $G$ is a connected graph of order $n \geq 8$ with $\delta(G) \geq 2$, then $\gamma(G) \leq 2n/5$, while Reed [7] showed

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that if $G$ is a graph of order $n$ with $\delta(G) \geq 3$, then $\gamma(G) \leq 3n/8$. As an application of Reed's result, we show that if $G$ is a graph of order $n \geq 14$ with $\delta(G) \geq 2$, then $\gamma(G) \leq \frac{3}{8}n + \frac{1}{8}k_1(G - \mathcal{L}) + \frac{1}{4}k_2(G - \mathcal{L})$.

**Keywords:** bounds, path-component, domination number
**AMS subject classification: 05C69**

# 1 Introduction

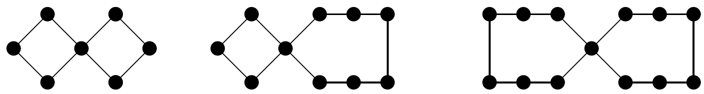
In this paper, we continue the study of domination in graphs. Domination in graphs is now well studied in graph theory. The literature on this subject has been surveyed and detailed in the two books by Haynes, Hedetniemi, and Slater [3, 4].

For notation and graph theory terminology we in general follow [3]. Specifically, let $G = (V, E)$ be a graph with vertex set $V$ of order $n = |V|$ and edge set $E$ of size $m = |E|$, and let $v$ be a vertex in $V$. The *open neighborhood* of $v$ is the set $N(v) = \{u \in V \mid uv \in E\}$ and the *closed neighbourhood* of $v$ is $N[v] = \{v\} \cup N(v)$. For a set $S$ of vertices, the open neighbourhood of $S$ is defined by $N(S) = \cup_{v \in S} N(v)$, and the closed neighbourhood of $S$ by $N[S] = N(S) \cup S$. If $X, Y \subseteq V$, then the set $X$ is said to *dominate* the set $Y$ if $Y \subseteq N[X]$. For a set $S \subseteq V$, the subgraph induced by $S$ is denoted by $G[S]$ while the graph $G - S$ is the graph obtained from $G$ by deleting the vertices in $S$ and all edges incident with $S$. We denote the degree of $v$ in $G$ by $d_G(v)$, or simply by $d(v)$ if the graph $G$ is clear from context. The minimum degree among the vertices of $G$ is denoted by $\delta(G)$.

We denote a path on $n$ vertices by $P_n$ and a cycle on $n$ vertices by $C_n$. We call a component of a graph a path-component if it is isomorphic to a path and a cycle-component if it is isomorphic to a cycle. A path-component isomorphic to a path $P_i$ we call a $P_i$-component, and a cycle-component isomorphic to a cycle $C_i$ we call a $C_i$-component.

We define a *daisy* to be a connected graph that can be constructed from two disjoint cycles by identifying a set of two vertices, one from each cycle, into one vertex. In particular, if the two cycles have lengths $n_1$ and $n_2$, we denote the daisy by $D(n_1, n_2)$. The daisies $D(4, 4)$, $D(4, 7)$ and $D(7, 7)$ are shown in Figure 1.

A *dominating set* of a graph $G = (V, E)$ is a set $S$ of vertices of $G$ such that every vertex $v \in V$ is either in $S$ or adjacent to a vertex of $S$. (That is, $N[S] = V$.) The *domination number* of $G$, denoted by $\gamma(G)$, is the minimum



$D(4,4)$ $D(4,7)$ $D(7,7)$

Figure 1: The daisies $D(4,4)$, $D(4,7)$ and $D(7,7)$.

cardinality of a dominating set. A dominating set of $G$ of cardinality $\gamma(G)$ is called a $\gamma(G)$-set. The domination number of a cycle or a path is easy to compute.

**Theorem 1** *For $n \geq 3$, $\gamma(P_n) = \gamma(C_n) = \lceil n/3 \rceil$.*

Let $G$ be a graph with $\delta(G) \geq 2$. We define a vertex as *small* if it has degree 2 and *large* if it has degree more than 2. Let $\mathcal{S}$ be the set of all small vertices of $G$ and $\mathcal{L}$ the set of all large vertices of $G$. Let $C$ be any component of $G - \mathcal{L}$. If $C$ is a component of $G$, then $C$ is a cycle; otherwise, if $C$ is not a component of $G$, then it is a path.

For $i \in \{0,1,2,3\}$, we denote the number of components of $G - \mathcal{L}$ of order congruent to $i$ modulo 4 by $p_i(G)$, or simply by $p_i$ if the graph $G$ is clear from context. If $G'$ is a graph, then for $i \in \{0,1,2,3\}$ we denote $p_i(G')$ simply by $p_i'$, and we denote the order and size of $G'$ by $n'$ and $m'$, respectively. Further, we denote the set of large vertices in $G'$ by $\mathcal{L}'$.

Let $\mathcal{B}_1 = \{C_4, C_7, D(4,4)\}$ and $\mathcal{B}_2 = \mathcal{B}_1 \cup \{C_{10}, C_{13}, D(4,7), D(7,7)\}$ be two families consisting of cycles and daisies. For $i = 1,2$, we say that a component is a $\mathcal{B}_i$-component if it is isomorphic to a graph in the family $\mathcal{B}_i$.

We call a component a *type-1 component* if it is a $P_i$-component for some $i \in \{3,4,5\}$ or a $C_5$-component, and we call a component a *type-2 component* if it is a $P_1$-component or a $P_2$-component. For $i = 1,2$, we denote the number of type-$i$ components in a graph $G$ by $k_i(G)$.

## 2 Known Results

The decision problem to determine the domination number of a graph is known to be NP-complete. Hence it is of interest to determine upper bounds on the domination number of a graph. Upper bounds have been established in [1, 2, 5, 6, 7, 8, 9] and elsewhere.

McCuaig and Shepherd [5] showed that the domination number of a connected graph with minimum degree at least 2 is at most two-fifths its order except for seven exceptional graphs (one of order four and six of order seven). More precisely, they defined a collection $\mathcal{B}$ of "bad" graphs shown in Figure 2, and proved the following result.

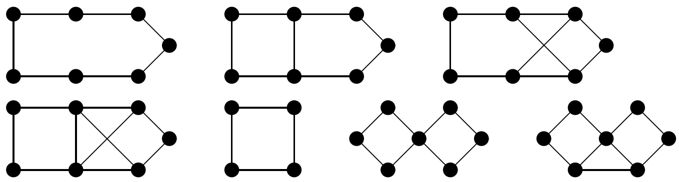


Figure 2: The family $\mathcal{B}$ of "bad" graphs.

**Theorem 2** (McCuaig and Shepherd [5]) *If $G$ is a connected graph of order $n$ with $\delta(G) \geq 2$ and $G \notin \mathcal{B}$, then $\gamma(G) \leq 2n/5$.*

In 1996, Reed [7] presented the important and useful result that if we restrict the minimum degree to be at least three, then the upper bound in Theorem 2 can be improved from two-fifths its order to three-eights its order.

**Theorem 3** (Reed [7]) *If $G$ is a graph of order $n$ with $\delta(G) \geq 3$, then $\gamma(G) \leq 3n/8$.*

## 3 Main Result

Our aim in the paper is to generalize Theorem 3 by relaxing the degree condition to minimum degree at least two. For notational convenience, for a graph $G$ of order $n$ and a graph $G'$ of order $n'$ we let

$$\begin{aligned}
\psi(G) &= \frac{3}{8}n + \frac{1}{8}(p_0 + p_3) + \frac{1}{4}(p_1 + p_2), \\
\psi(G') &= \frac{3}{8}n' + \frac{1}{8}(p_0' + p_3') + \frac{1}{4}(p_1' + p_2'), \\
\varphi(G) &= \frac{3}{8}n + \frac{1}{8}k_1(G - \mathcal{L}) + \frac{1}{4}k_2(G - \mathcal{L}), \text{ and} \\
\varphi(G') &= \frac{3}{8}n' + \frac{1}{8}k_1(G' - \mathcal{L}') + \frac{1}{4}k_2(G' - \mathcal{L}').
\end{aligned}$$

We shall prove:

**Theorem 4** *If $G$ is a graph of order $n$ with $\delta(G) \geq 2$ that has no $\mathcal{B}_1$-component, then $\gamma(G) \leq \psi(G)$.*

**Theorem 5** *If $G$ is a graph of order $n$ with $\delta(G) \geq 2$ that has no $\mathcal{B}_2$-component, then $\gamma(G) \leq \varphi(G)$.*

### 3.1 Preliminary Observations

Let $G$ be an arbitrary graph. By *attaching a $G_8$-unit* to a specified vertex $v$ of $G$, we mean adding a (disjoint) copy of the graph $G_8$ of Figure 3 and identifying any one of its vertices that is in a triangle with $v$.

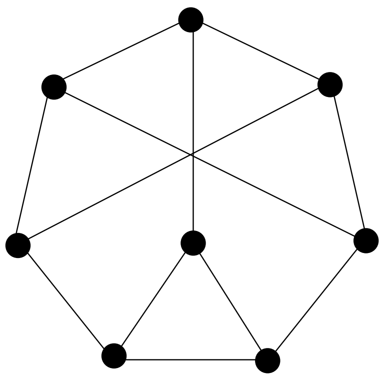


Figure 3: A cubic graph $G_8$ with domination number 3

We will frequently use the following observation in the proof of Theorem 4.

**Observation 1** *If $G'$ is obtained from a graph $G$ by attaching a $G_8$-unit to a vertex $v$, then there exists a $\gamma(G')$-set that contains $v$ and two other vertices in the $G_8$-unit.*

We define an *elementary 3-subdivision* of a nonempty graph $G$ as a graph obtained from $G$ by subdividing some edge three times. The following observation will prove to be useful.

**Observation 2** *If $G$ is obtained from a nontrivial graph $G'$ by an elementary 3-subdivision, then $\gamma(G) = \gamma(G') + 1$.*

We will refer to a graph $G$ as a *reduced graph* if $G$ has no induced path on five vertices, the internal vertices of which have degree 2 in $G$. Hence if $u, v_1, v_2, v_3, v$ is a path in a reduced graph $G$, then $d_G(v_i) \geq 3$ for at least one $i$, $1 \leq i \leq 3$, or $uv \in E(G)$.

### 3.2 Proof of Theorem 4

It suffices to prove that if $G$ is a *connected* graph of order $n$ with $\delta(G) \geq 2$ and $G \notin \mathcal{B}_1$, then $\gamma(G) \leq \psi(G)$. We proceed by induction on the order of the lexicographic sequence $(p_0 + p_1 + p_2 + p_3, n, m)$, where $p_0 + p_1 + p_2 + p_3 \geq 0$, $n \geq 3$ and $m \geq 3$. We remark that the order of the considered graphs does not always have to drop when applying an inductive argument. For notational convenience, for a graph $G$ of order $n$ and size $m$ and a graph $G'$ of order $n'$ and size $m'$, we denote the sequence $(p_0 + p_1 + p_2 + p_3, n, m)$ by $s(G)$ and the sequence $(p'_0 + p'_1 + p'_2 + p'_3, n', m')$ by $s(G')$. Further, we denote the set of small vertices of $G$ and $G'$ by $\mathcal{S}$ and $\mathcal{S}'$, respectively, and the set of large vertices of $G$ and $G'$ by $\mathcal{L}$ and $\mathcal{L}'$, respectively.

When $p_0 + p_1 + p_2 + p_3 = 0$, the graph $G$ has only large vertices. Thus, $\delta(G) \geq 3$ and the desired result follows from Theorem 3. This establishes the base case. Let $p_0 + p_1 + p_2 + p_3 \geq 1$, $n \geq 3$ and $m \geq 3$. Assume that for all connected graphs $G' \notin \mathcal{B}_1$ of order $n'$ with $\delta(G') \geq 2$ that have lexicographic sequence $s(G')$ smaller than $s$, $\gamma(G') \leq \psi(G')$. Let $G \notin \mathcal{B}_1$ be a connected graph of order $n$, size $m$ with $\delta(G) \geq 2$ and with lexicographic sequence $s(G) = s$. Let $G = (V, E)$. We proceed further with a series of claims that we may assume the graph $G$ satisfies.

**Claim A** *$G$ is a reduced graph.*

**Proof.** Assume that $G$ is not a reduced graph. Then, $G$ contains an induced path $u, v_1, v_2, v_3, v$ on five vertices, the internal vertices of which have degree 2 in $G$ and $uv \notin E$ (possibly, $u$ or $v$ or both $u$ and $v$ are large vertices in $G$). Let $G' = (G - \{v_1, v_2, v_3\}) \cup \{uv\}$. Then, $\delta(G') \geq 2$ and $G$ is obtained from $G'$ by an elementary 3-subdivision. By Observation 2, $\gamma(G) = \gamma(G') + 1$. If $G' = C_4$, then $G = C_7$ and $G \in \mathcal{B}_1$, a contradiction. If $G' = C_7$, then $G = C_{10}$, while if $G' = D(4,4)$, then $G = D(4,7)$. In both cases, $\gamma(G) = 4 = \psi(G)$, and the desired bound holds. Hence we may assume that $G' \notin \mathcal{B}_1$. Since $p'_0+p'_1+p'_2+p'_3 \leq p_0+p_1+p_2+p_3$ and $n' = n-3$, the lexicographic sequence $s(G')$ is smaller than $s(G)$. Applying the inductive hypothesis to $G'$, $\gamma(G') \leq \psi(G')$. Hence, $\gamma(G) - 1 = \gamma(G') \leq \psi(G') \leq \psi(G) - (3/8) * 3 - 1/8 + 1/4 = \psi(G) - 1$, and so $\gamma(G) \leq \psi(G)$. Hence we may assume that $G$ is a reduced graph. □

**Claim B** *$G$ is not a cycle.*

**Proof.** Assume that $G$ is a cycle. By Claim A, either $G = C_3$ or $G = C_5$. On the one hand, if $G = C_3$, then $\gamma(G) = 1 < (3/8) * 3 + 1/8 = \psi(G)$. On the other hand, if $G = C_5$, then $\gamma(G) = 2 < (3/8) * 5 + 1/4 = \psi(G)$. In both cases, $\gamma(G) < \psi(G)$. $\Box$

Note that if $G - \mathcal{L}$ has a cycle-component $C$, then $C$ is also a cycle-component of $G$, implying that $G = C$ since $G$ is connected. Hence by Claim B, every component of $G - \mathcal{L}$ is a path-component. By Claim A, every path-component has order 1, 2, 3 or 4.

**Claim C** $p_0 = 0$.

**Proof.** Suppose that $p_0 \geq 1$. Let $P\colon v_1, v_2, v_3, v_4$ be a $P_4$-component of $G[\mathcal{S}]$. Since $G$ is a reduced graph, the two ends of $P$ are adjacent in $G$ to the same large vertex. Let $v$ be the common large neighbor of $v_1$ and $v_4$. Then, $v, v_1, v_2, v_3, v_4, v$ is a cycle in $G$. Let $G'$ be the graph obtained from $G - V(P)$ by attaching a $G_8$-unit to the vertex $v$. Then, $G'$ is a graph of order $n' = n+3$ with $\delta(G') \geq 2$. Since $G \notin \mathcal{B}_1$, we have that $G' \notin \mathcal{B}_1$. Further $p'_0+p'_1+p'_2+p'_3 = p_0+p_1+p_2+p_3-1$, and so the lexicographic sequence $s(G')$ is smaller than $s(G)$. Applying the inductive hypothesis to $G'$, $\gamma(G') \leq \psi(G')$. By Observation 1, there exists a $\gamma(G')$-set $D'$ that contains $v$ and a set $D_v$ of two other vertices in the attached $G_8$-unit. Hence, $D = (D' \setminus D_v) \cup \{v_2\}$ is a dominating set in $G$. Thus, $\gamma(G) \leq |D| = |D'| - 1 = \gamma(G') - 1$. Therefore, $\gamma(G) + 1 \leq \gamma(G') \leq \psi(G') \leq \psi(G) + (3/8) * 3 - 1/8 = \psi(G) + 1$. Consequently, $\gamma(G) \leq \psi(G)$. $\Box$

**Claim D** $p_3 = 0$.

**Proof.** Suppose that $p_3 \geq 1$. Let $P\colon v_1, v_2, v_3$ be a $P_3$-component of $G[\mathcal{S}]$. Let $u$ be the neighbor of $v_1$ not on $P$ and let $v$ be the neighbor of $v_3$ not on $P$. We consider two possibilities.

*Case 1.* $u = v$. Then, $v, v_1, v_2, v_3, v$ is a cycle in $G$. Suppose $d_G(v) \geq 4$. Let $G' = G - V(P)$. Then, $\delta(G') \geq 2$. If $G' = C_4$, then $G = D(4,4)$ and $G \in \mathcal{B}_1$, a contradiction. If $G' = C_7$ or if $G' = D(4,4)$, then $\gamma(G) = 4 = (3/8) * 7 + 3/8 \leq \psi(G)$, and the desired bound holds. Hence we may assume that $G' \notin \mathcal{B}_1$. Since $p'_0+p'_1+p'_2+p'_3 \leq p_0+p_1+p_2+p_3$ and $n' = n-3$, the lexicographic sequence $s(G')$ is smaller than $s(G)$. Applying the inductive hypothesis to $G'$, $\gamma(G') \leq \psi(G')$. Every $\gamma(G')$-set can be extended to a dominating set of $G$ by adding to it the vertex $v_2$, and so $\gamma(G) \leq \gamma(G') + 1$. Hence, $\gamma(G) - 1 \leq \gamma(G') \leq \psi(G') \leq \psi(G) - (3/8) * 3 - 1/8 + 1/4 = \psi(G) - 1$, and so $\gamma(G) \leq \psi(G)$. Hence we may assume that $d_G(v) = 3$.

Let $w$ be the neighbor of $v$ not on $P$. If $d_G(w) = 2$, let $x$ be the neighbor of $w$ different from $v$. If $d_G(x) = 2$, let $y$ be the neighbor of $x$ different from $w$.

Let $G'$ be the graph obtained from $G - \{v, v_1, v_2, v_3\}$ by attaching a $G_8$-unit to the vertex $w$. Then, $G'$ is a graph of order $n' = n + 3$ with $\delta(G') \geq 2$. Since $G \notin \mathcal{B}_1$, we have that $G' \notin \mathcal{B}_1$.

If $d_G(w) \geq 3$, then $p'_3 = p_3 - 1$ and $p'_i = p_i$ for $i \in \{0, 1, 2\}$. If $d_G(w) = 2$ and $d_G(x) \geq 3$, then, $p'_1 = p_1 - 1$, $p'_3 = p_3 - 1$ and $p'_i = p_i$ for $i \in \{0, 2\}$. If $d_G(w) = d_G(x) = 2$, then since $G$ is a reduced graph, we have that $d_G(y) \geq 3$, and so $p'_0 = p_0$, $p'_1 = p_1 + 1$, $p'_2 = p_2 - 1$, and $p'_3 = p_3 - 1$. Therefore in all three cases, $p'_0 + p'_1 + p'_2 + p'_3 \leq p_0 + p_1 + p_2 + p_3 - 1$, and so the lexicographic sequence $s(G')$ is smaller than $s(G)$. Further, $\psi(G') \leq \psi(G) + (3/8) * 3 - 1/8 = \psi(G) + 1$.

Applying the inductive hypothesis to $G'$, $\gamma(G') \leq \psi(G')$. By Observation 1, there exists a $\gamma(G')$-set $D'$ that contains $w$ and a set $D_w$ of two other vertices in the attached $G_8$-unit. Hence, $D = (D' \setminus D_w) \cup \{v_2\}$ is a dominating set in $G$. Thus, $\gamma(G) \leq |D| = |D'| - 1 = \gamma(G') - 1$. Consequently, $\gamma(G) + 1 \leq \gamma(G') \leq \psi(G') \leq \psi(G) + 1$, whence $\gamma(G) \leq \psi(G)$.

*Case 2.* $u \neq v$. Since $G$ is a reduced graph, we must have $uv \in E$. Let $G' = G - V(P)$. Then, $\delta(G') \geq 2$. If $G' = C_4$, then $\gamma(G) = 3$. Further $n = 7$, and $p_2 = p_3 = 1$ and $p_0 = p_1 = 0$, and so $\psi(G) = (3/8) * 7 + 1/8 + 1/4 = 3$. Thus if $G' = C_4$, then $\gamma(G) = \psi(G)$. If $G' = C_7$, then $G$ would not be a reduced graph, contrary to assumption. If $G' = D(4, 4)$, then $\gamma(G) = 4$. Further $n = 10$, and $p_3 = 2$ and $p_1 + p_2 \geq 1$, and so $\psi(G) = (3/8) * 10 + 2/8 + 1/4 > 4$. Thus, if $G' = D(4, 4)$, then $\gamma(G) < \psi(G)$. Hence we may assume that $G' \notin \mathcal{B}_1$. Since $p'_0 + p'_1 + p'_2 + p'_3 \leq p_0 + p_1 + p_2 + p_3$ and $n' = n - 3$, the lexicographic sequence $s(G')$ is smaller than $s(G)$. Applying the inductive hypothesis to $G'$, $\gamma(G') \leq \psi(G')$. Every $\gamma(G')$-set can be extended to a dominating set of $G$ by adding to it the vertex $v_2$, and so $\gamma(G) \leq \gamma(G') + 1$. Hence, $\gamma(G) - 1 \leq \gamma(G') \leq \psi(G') \leq \psi(G) - (3/8) * 3 - 1/8 + 1/4 = \psi(G) - 1$, and so $\gamma(G) \leq \psi(G)$. □

**Claim E** $p_2 = 0$.

**Proof.** Suppose that $p_2 \geq 1$. Let $P\colon v_1, v_2$ be a $P_2$-component of $G[\mathcal{S}]$. Let $u$ be the neighbor of $v_1$ not on $P$ and let $v$ be the neighbor of $v_2$ not on $P$.

If the one hand, suppose that $u = v$. Let $G'$ be the graph obtained from $G - V(P)$ by attaching a $G_8$-unit to the vertex $v$. Then, $G'$ is a graph of order $n' = n + 5$ with $\delta(G') \geq 2$. Since $G \notin \mathcal{B}_1$, we have that $G' \notin \mathcal{B}_1$. Further, $p'_2 = p_2 - 1$ and $p'_i = p_i$ for $i \in \{0, 1, 3\}$. Hence, $p'_0 + p'_1 + p'_2 + p'_3 = p_0 + p_1 + p_2 + p_3 - 1$, and so the lexicographic sequence $s(G')$ is smaller than $s(G)$. Applying the inductive hypothesis to $G'$, $\gamma(G') \leq \psi(G')$. By Observation 1, there exists a $\gamma(G')$-set $D'$ that contains $v$ and a set $D_v$ of two other vertices in the attached $G_8$-unit. Hence, $D = D' \setminus D_v$ is a dominating set in $G$. Thus,

$\gamma(G) \le |D| = |D'| - 2 = \gamma(G') - 2$. Therefore, $\gamma(G) + 2 \le \gamma(G') \le \psi(G') \le \psi(G) + (3/8) * 5 - 1/4 < \psi(G) + 2$. Consequently, $\gamma(G) \le \psi(G)$.

If the other hand, suppose that $u \ne v$. If $uv \in E$, then let $G' = G - uv$. Then, $\delta(G') \ge 2$. By our structure of $G$, $G' \notin \{C_4, D(4,4)\}$. If $G' = C_7$, then $p_3 = 1$, contrary to our assumption in Claim D. Hence, $G' \notin \mathcal{B}_1$. Further, $p'_0 + p'_1 + p'_2 + p'_3 = p_0 + p_1 + p_2 + p_3$. Thus since $G'$ has order $n' = n$ and size $m' = m-1$, the lexicographic sequence $s(G')$ is smaller than $s(G)$. Applying the inductive hypothesis to $G'$, $\gamma(G') \le \psi(G') \le \psi(G)$. Since the domination number of a graph cannot decrease if edges are removed, $\gamma(G) \le \gamma(G')$, implying that $\gamma(G) \le \psi(G)$. Hence we may assume that $uv \notin E$.

Let $G'$ be obtained from $G - V(P)$ by adding the edge $uv$. Then, $\delta(G') \ge 2$ and both $u$ and $v$ are large vertices in $G'$. Since $G \notin \mathcal{B}_1$, we have that $G' \notin \mathcal{B}_1$. Further $p'_2 = p_2 - 1$ while $p'_i = p_i$ for $i \in \{0,1,3\}$. Thus since $p'_0 + p'_1 + p'_2 + p'_3 = p_0 + p_1 + p_2 + p_3 - 1$, the lexicographic sequence $s(G')$ is smaller than $s(G)$. Applying the inductive hypothesis to $G'$, $\gamma(G') \le \psi(G')$. Every $\gamma(G')$-set can be extended to a dominating set of $G$ by adding to it $v_1$ or $v_2$, and so $\gamma(G) \le \gamma(G') + 1$. Therefore, $\gamma(G) - 1 \le \gamma(G') \le \psi(G') \le \psi(G) - (3/8) * 2 - 1/4 = \psi(G) - 1$. Consequently, $\gamma(G) \le \psi(G)$. $\square$

By Claims C, D and E, we have $p_0 = p_2 = p_3 = 0$ and $p_1 \ge 1$. Thus, by our earlier assumptions, every component of $G[\mathcal{S}] = G - \mathcal{L}$ is a $P_1$-component. Let $P$ be a $P_1$-component of $G[\mathcal{S}]$ with $V(P) = \{v_1\}$. Let $u$ and $v$ be the two neighbors of $v_1$. Then, $\{u, v\} \subseteq \mathcal{L}$.

**Claim F** *$uv \notin E$.*

**Proof.** Suppose that $uv \in E$. Let $G' = G - uv$. Then, $\delta(G') \ge 2$ and $\gamma(G) \le \gamma(G')$. If $G' = C_4$, then $\gamma(G) = 1 < \psi(G)$. Since $G$ is a reduced graph, $G' \ne C_7$. If $G' = D(4,4)$, then $n = 7$ and $\gamma(G) = 2 < \psi(G)$. Hence, we may assume that $G' \notin \mathcal{B}_1$. Further, $p'_0 + p'_1 + p'_2 + p'_3 \le p_0 + p_1 + p_2 + p_3$. Thus since $G'$ has order $n' = n$ and size $m' = m - 1$, the lexicographic sequence $s(G')$ is smaller than $s(G)$. Applying the inductive hypothesis to $G'$, $\gamma(G') \le \psi(G')$. Since $\gamma(G) \le \gamma(G')$ and $\psi(G') \le \psi(G)$, we have that $\gamma(G) \le \psi(G)$. $\square$

**Claim G** *The vertices $u$ and $v$ have only one common degree-2 neighbor.*

**Proof.** Suppose that $u$ and $v$ have a common degree-2 neighbor $v_2$ that is different from $v_1$. Let $G'$ be obtained from $G - \{v_1, v_2\}$ by adding the edge $uv$. Then, $\delta(G') \ge 2$ and $\gamma(G) \le \gamma(G') + 1$. If $G' = C_4$, then $n = 6$ and

$\gamma(G) = 2 < \psi(G)$. Since $G$ is a reduced graph, $G' \neq C_7$. If $G' = D(4,4)$, then $n = 9$ and $\gamma(G) = 3 < \psi(G)$. Hence, we may assume that $G' \notin \mathcal{B}_1$. Further, $p'_0 + p'_1 + p'_2 + p'_3 \leq p_0 + p_1 + p_2 + p_3 - 1$, and so the lexicographic sequence $s(G')$ is smaller than $s(G)$. Applying the inductive hypothesis to $G'$, $\gamma(G') \leq \psi(G')$. Hence, $\gamma(G) - 1 \leq \gamma(G') \leq \psi(G') \leq \psi(G) - (3/8)*2 - (1/4)*2 + 1/4 = \psi(G) - 1$. Thus, $\gamma(G) \leq \psi(G)$. $\Box$

**Claim H** *The vertices $u$ and $v$ have at least one common neighbor different from $v_1$, and each such common neighbor is a degree-3 vertex in $G$.*

**Proof.** Suppose that $v_1$ is the only common neighbor of $u$ and $v$. Let $G'$ be obtained from $G - \{u, v, v_1\}$ by adding a new vertex $w$ and joining it to all vertices in $(N(u) \cup N(v)) \setminus \{v_1\}$. Then, $d_{G'}(w) \geq 4$, $\delta(G') \geq 2$ and $\gamma(G) \leq \gamma(G') + 1$. If $G' \in \mathcal{B}_1$ then $G' = D(4,4)$, $n = 9$, and $\gamma(G) = 3 < \psi(G)$. Hence $G' \notin \mathcal{B}_1$. Further, $p'_1 = p_1 - 1$ and $p'_i = p_i$ for $i \in \{0, 2, 3\}$. Thus, $p'_0 + p'_1 + p'_2 + p'_3 = p_0 + p_1 + p_2 + p_3 - 1$, and so the lexicographic sequence $s(G')$ is smaller than $s(G)$. Applying the inductive hypothesis to $G'$, $\gamma(G') \leq \psi(G')$. Hence, $\gamma(G) - 1 \leq \gamma(G') \leq \psi(G') \leq \psi(G) - (3/8)*2 - 1/4 = \psi(G) - 1$. Consequently, $\gamma(G) \leq \psi(G)$.

Now suppose that $w$ is a common neighbor of $u$ and $v$ different from $v_1$. Suppose that $d_G(w) \geq 4$. Let $G' = G - vw$. Then, $\delta(G') \geq 2$. Further, $G' \notin \mathcal{B}_1$ and $p'_0 + p'_1 + p'_2 + p'_3 \leq p_0 + p_1 + p_2 + p_3$. Thus since $G'$ has order $n' = n$ and size $m' = m - 1$, the lexicographic sequence $s(G')$ is smaller than $s(G)$. Applying the inductive hypothesis to $G'$, $\gamma(G') \leq \psi(G')$. Since $\gamma(G) \leq \gamma(G')$ and $\psi(G') \leq \psi(G)$, we have that $\gamma(G) \leq \psi(G)$. Hence we may assume that $d_G(w) \leq 3$. By Claim G, $d_G(w) \geq 3$. Consequently, $d_G(w) = 3$. $\Box$

**Claim I** *Both $u$ and $v$ are degree-3 vertices in $G$.*

**Proof.** Suppose that $u$ or $v$ has degree greater than 3. Without loss of generality, we may assume that $d_G(u) \geq 4$. Let $G'$ be the graph obtained from $G - v_1$ by attaching a $G_8$-unit to the vertex $v$. Then, $G'$ is a graph of order $n' = n + 6$ with $\delta(G') \geq 2$. Note that both $u$ and $v$ are large vertices in $G'$. Since $n' > 7$, we have that $G' \notin \mathcal{B}_1$. Further, $p'_1 = p_1 - 1$ and $p'_i = p_i$ for $i \in \{0, 2, 3\}$. Thus, $p'_0 + p'_1 + p'_2 + p'_3 = p_0 + p_1 + p_2 + p_3 - 1$, and so the lexicographic sequence $s(G')$ is smaller than $s(G)$. Applying the inductive hypothesis to $G'$, $\gamma(G') \leq \psi(G')$. By Observation 1, there exists a $\gamma(G')$-set $D'$ that contains $v$ and a set $D_v$ of two other vertices in the attached $G_8$-unit. Hence, $D = D' \setminus D_v$ is a dominating set in $G$. Thus, $\gamma(G) \leq |D| = |D'| - 2 = \gamma(G') - 2$. Therefore,

$\gamma(G) + 2 \le \gamma(G') \le \psi(G') \le \psi(G) + (3/8) * 6 - 1/4 = \psi(G) + 2$. Consequently, $\gamma(G) \le \psi(G)$. $\Box$

By Claim H, we may assume that there is a degree-3 vertex $y$ that is adjacent to both $u$ and $v$. By Claim I, we may assume that both $u$ and $v$ are degree-3 vertices in $G$. Let $N(u) = \{v_1, y, w\}$ and let $N(v) = \{v_1, y, z\}$.

**Claim J** *$w = z$.*

**Proof.** Suppose that $w \neq z$.

Since $d_G(y) = 3$ and $\{u, v\} \subset N(y)$, the vertex $y$ is adjacent to at most one of $w$ and $z$. Without loss of generality, we may assume that $yz \notin E$. Let $G'$ be obtained from $G - \{v, v_1\}$ by adding the two edges $uz$ and $yz$. Then, $\delta(G') \ge 2$ and each of $u$, $y$ and $z$ is a large vertex in $G'$. Let $D'$ be a $\gamma(G')$-set. If $z \in D'$, then $D' \cup \{u\}$ is a dominating set of $G$, while if $z \notin D'$, then $D' \cup \{v\}$ is a dominating set of $G$. Hence every $\gamma(G')$-set can be extended to a dominating set of $G$ by adding to it either $u$ or $v$. Thus, $\gamma(G) \le \gamma(G') + 1$. Since $G \notin \mathcal{B}_1$, we have that $G' \notin \mathcal{B}_1$. Further, $p'_1 \le p_1 - 1$ and $p'_i \le p_i$ for $i \in \{0, 2, 3\}$. Thus, $p'_0 + p'_1 + p'_2 + p'_3 \le p_0 + p_1 + p_2 + p_3 - 1$, and so the lexicographic sequence $s(G')$ is smaller than $s(G)$. Applying the inductive hypothesis to $G'$, $\gamma(G') \le \psi(G')$. Hence, $\gamma(G) - 1 \le \gamma(G') \le \psi(G') \le \psi(G) - (3/8) * 2 - 1/4 = \psi(G) - 1$. Consequently, $\gamma(G) \le \psi(G)$. $\Box$

By Claim J, we may assume that $w = z$, and so $w$ is a common neighbor of $u$ and $v$ different from $v_1$. By Claim H, $d_G(w) = 3$. Let $G' = G - uy - vw$. Then, $\delta(G') \ge 2$. If $G' \in \mathcal{B}_1$, then $G' = C_7$. But then $G - \mathcal{L}$ would contain a $P_2$-component, contrary to our earlier assumption. Hence, $G' \notin \mathcal{B}_1$. Further, $p'_0 + p'_1 + p'_2 + p'_3 \le p_0 + p_1 + p_2 + p_3$. Thus since $G'$ has order $n' = n$ and size $m' = m-2$, the lexicographic sequence $s(G')$ is smaller than $s(G)$. Applying the inductive hypothesis to $G'$, $\gamma(G') \le \psi(G')$. Since $\gamma(G) \le \gamma(G')$ and $\psi(G') \le \psi(G)$, we have that $\gamma(G) \le \psi(G)$. This completes the proof of Theorem 4. $\Box$

### 3.3 Proof of Theorem 5

Assume the theorem is false. Among all counterexamples, let $G$ be one of minimum order $n$. Then, $G$ is a connected graph with $\delta(G) \ge 2$, $G \notin \mathcal{B}_2$, and $\gamma(G) > \varphi(G)$. We proceed further with three claims.

**Claim K** *The graph $G - \mathcal{L}$ has no cycle-component.*

**Proof.** Assume, to the contrary, that $G-\mathcal{L}$ has a cycle-component $C$. Then, $C$ is also a cycle-component of $G$, implying that $G = C = C_n$ and $\gamma(G) = \lceil n/3 \rceil$. Since $G \notin \mathcal{B}_2$, $n \notin \{4, 7, 10, 13\}$. If $n \equiv 0 \,(\mathrm{mod}\, 3)$, then $\gamma(G) = n/3 < 3n/8 \le \varphi(G)$. If $n \equiv 1 \,(\mathrm{mod}\, 3)$, then $n \ge 16$ and $\gamma(G) = (n+2)/3 \le 3n/8 = \varphi(G)$. If $n \equiv 2 \,(\mathrm{mod}\, 3)$, then either $n = 5$ and $\gamma(G) = 2 = 3n/8 + 1/8 = \varphi(G)$ or $n \ge 8$ and $\gamma(G) = (n+1)/3 \le 3n/8 = \varphi(G)$. In all three cases, $\gamma(G) \le \varphi(G)$, contradicting our assumption that $G$ is a counterexample to Theorem 5. □

By Claim K, the graph $G - \mathcal{L}$ has no cycle-component. Thus, $|\mathcal{L}| \ge 1$ and every component of $G - \mathcal{L}$ is a path-component.

**Claim L** *The graph $G - \mathcal{L}$ has no path-component of order $k \ge 8$.*

**Proof.** Assume, to the contrary, that $P \colon v_1, v_2, \dots, v_k$ is a $P_k$-component of $G-\mathcal{L}$ where $k \ge 8$. Let $u$ be the neighbor of $v_1$ not on $P$ and let $v$ be the neighbor of $v_k$ not on $P$. (Possibly, $u = v$.) Let $G' = (G - \{v_1, v_2, \dots, v_6\}) \cup \{uv_7\}$ and let $P' = P - \{v_1, v_2, \dots, v_6\}$. Then, $G'$ is a connected graph of order $n' = n - 6$ with $\delta(G') \ge 2$. It follows from Observation 2 that $\gamma(G) = \gamma(G') + 2$. Note that the set of large vertices of $G'$ is the set $\mathcal{L}$.

If $G' \in \mathcal{B}_2$, then $G' \in \{D(4,4), D(4,7), D(7,7)\}$. Since $G \notin \mathcal{B}_2$, this implies that $G \in \{D(4,10), D(4,13), D(7,10), D(7,13)\}$. In all cases, $\gamma(G) \le \varphi(G)$, a contradiction. Hence, $G' \notin \mathcal{B}_2$. Since $G'$ is not a counterexample to our theorem, $\gamma(G') \le \varphi(G')$.

Note that the type-1 or type-2 components of $G' - \mathcal{L}$ and $G - \mathcal{L}$ are the same, except that $G' - \mathcal{L}$ may contain one additional type-1 or type-2 component, namely the component $P'$. Hence, $\varphi(G') \le \varphi(G) - (3/8) * 6 + 1/4 = \varphi(G) - 2$. Thus, $\gamma(G) = \gamma(G') + 2 \le \varphi(G') + 2 \le \varphi(G)$, a contradiction. □

**Claim M** *The graph $G - \mathcal{L}$ has no path-component of order 5, 6 or 7.*

**Proof.** Assume, to the contrary, that $P \colon v_1, v_2, \dots, v_k$ is a $P_k$-component of $G-\mathcal{L}$, where $k \in \{5, 6, 7\}$. Let $u$ be the neighbor of $v_1$ not on $P$ and let $v$ be the neighbor of $v_k$ not on $P$. (Possibly, $u = v$.) Let $G' = (G - \{v_1, v_2, v_3\}) \cup \{uv_4\}$ and let $P' = P - \{v_1, v_2, v_3\}$. Then, $G'$ is a connected graph of order $n' = n - 3$ with $\delta(G') \ge 2$. By Observation 2, $\gamma(G) = \gamma(G') + 1$. Note that the set of large vertices of $G'$ is the set $\mathcal{L}$.

Suppose $k = 5$. Since $G$ has no $\mathcal{B}_2$-component, neither does $G'$. Note that the type-1 or type-2 components of $G' - \mathcal{L}$ and $G - \mathcal{L}$ are the same, except for the type-1 component $P$ of $G - \mathcal{L}$ which becomes the type-2 component $P'$ of

$G'-\mathcal{L}$. Hence, $k_1(G'-\mathcal{L}) = k_1(G-\mathcal{L})-1$ and $k_2(G'-\mathcal{L}) = k_2(G-\mathcal{L})+1$, and so $\varphi(G') = \varphi(G)-(3/8)*3-1/8+1/4 = \varphi(G)-1$. Since $G'$ is not a counterexample to our theorem, $\gamma(G') \le \varphi(G')$. Hence, $\gamma(G) = \gamma(G')+1 \le \varphi(G')+1 = \varphi(G)$, a contradiction.

Suppose $k \in \{6,7\}$. If $G' \in \mathcal{B}_2$, then $k = 6$ and $G \in \{D(4,10), D(7,10)\}$ and $\gamma(G) \le \varphi(G)$, a contradiction. Hence, $G' \notin \mathcal{B}_2$. Note that the type-1 or type-2 components of $G'-\mathcal{L}$ and $G-\mathcal{L}$ are the same, except that $G'-\mathcal{L}$ contains one additional type-1 component, namely the component $P'$. Hence, $k_1(G'-\mathcal{L}) = k_1(G-\mathcal{L})+1$ and $k_2(G'-\mathcal{L}) = k_2(G-\mathcal{L})$, and so $\varphi(G') = \varphi(G)-(3/8)*3+1/8 = \varphi(G)-1$. Since $G'$ is not a counterexample to our theorem, $\gamma(G') \le \varphi(G')$. Hence, $\gamma(G) = \gamma(G')+1 \le \varphi(G')+1 = \varphi(G)$, a contradiction. $\square$

By Claims L and M, every path-component of $G-\mathcal{L}$ has order at most 4. Hence, $k_1(G-\mathcal{L}) = p_0+p_3$ and $k_2(G-\mathcal{L}) = p_1+p_2$, and so $\psi(G) = \varphi(G)$. Thus, by Theorem 4, $\gamma(G) \le \varphi(G)$, a contradiction. This completes the proof of Theorem 5. $\square$

That the bound of Theorem 5 is in a sense best possible, may be seen as follows. Let $v$ be a specified vertex of some graph. By *attaching a $C_n$-unit* to $v$, we mean adding a (disjoint) copy of an $n$-cycle and identifying any one of its vertices with $v$. By *attaching a key-unit* to $v$, we mean adding a (disjoint) copy of a 4-cycle and joining with an edge one of its vertices to $v$. Let $\mathcal{G}$ denote the family of all graphs that can be obtained from a connected graph $F$ by attaching to each vertex $v$ of $F$ a $G_8$-unit, a $C_5$-unit, a $C_8$-unit, or if $d_F(v) \ge 2$, a key-unit. A graph in the family $\mathcal{G}$ with one key-unit, one $C_5$-unit and one $G_8$-unit that is obtained from a complete graph $F = K_3$ on three vertices is illustrated in Figure 4.

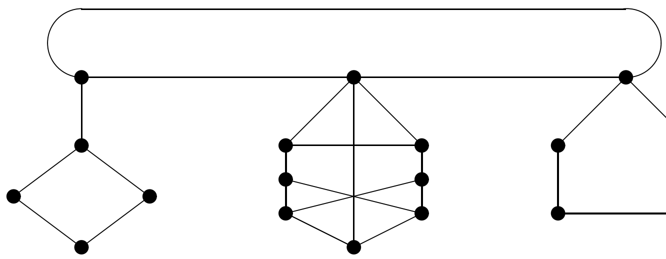


Figure 4: A graph in the family $\mathcal{G}$.

If $G \in \mathcal{G}$, then each key-unit and each $C_5$-unit of $G$ contributes two to $\gamma(G)$, five to $|V(G)|$, one to $k_1(G-\mathcal{L})$, and zero to $k_2(G-\mathcal{L})$, while each $C_8$-unit and each $G_8$-unit contributes three to $\gamma(G)$, eight to $|V(G)|$ and zero to both

$k_1(G-\mathcal{L})$ and $k_2(G-\mathcal{L})$. Thus, if $G \in \mathcal{G}$ has order $n$ with $a$ key-unit, $b$ $C_5$-units, $c$ $C_8$-units, and $d$ $G_8$-units, then $n = 5(a+b)+8(c+d)$, $k_1(G-\mathcal{L}) = a+b$ and $\gamma(G) = 2(a+b)+3(c+d) = \psi(G)$.

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# References

[1] W. Duckworth, N. C. Wormald, Minimum independent dominating sets of random cubic graphs. *Random Struct. Alg.* 21 (2002), 147–161.

[2] W. Duckworth, N. C. Wormald, On the independent domination number of random regular graphs. *Combin. Probab. Comput.* **15** (2006), 513–522.

[3] T.W. Haynes, S.T. Hedetniemi, and P.J. Slater (eds.), *Fundamentals of Domination in Graphs*, Marcel Dekker, Inc. New York, 1998.

[4] T.W. Haynes, S.T. Hedetniemi, and P.J. Slater (eds.), *Domination in Graphs: Advanced Topics*, Marcel Dekker, Inc. New York, 1998.

[5] W. McCuaig and B. Shepherd, Domination in graphs with minimum degree two. *J. Graph Theory* **13** (1989), 749–762.

[6] M. Molloy, and B. Reed, The dominating number of a random cubic graph. *Combin. Probab. Comput.* **7** (1995), 209–221.

[7] B. A. Reed, Paths, stars and the number three. *Combin. Probab. Comput.* **5** (1996), 277–295.

[8] L. A. Sanchis, Bounds related to domination in graphs with minimum degree two. *J. Graph Theory* **25** (1997), 139–152.

[9] H. M. Xing, L. Sun, X. G. Chen, Domination in graphs of minimum degree five. *Graphs Combin.* **22** (2006), 127–143.

# On a Conjecture about Inverse Domination in Graphs

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**Abstract**

Let $G = (V, E)$ be a graph with no isolated vertex. A classical observation in domination theory is that if $D$ is a minimum dominating set of $G$, then $V \setminus D$ is also a dominating set of $G$. A set $D'$ is an inverse dominating set of $G$ if $D'$ is a dominating set of $G$ and $D' \subseteq V \setminus D$ for some minimum dominating set $D$ of $G$. The inverse domination number of $G$ is the minimum cardinality among all inverse dominating sets of $G$. The independence number of $G$ is the maximum cardinality of an independent set of vertices in $G$. Domke, Dunbar, and Markus (Ars Combin. 72 (2004), 149–160) conjectured that the inverse domination number of $G$ is at most the independence number of $G$. We prove this conjecture for special families of graphs, including claw-free graphs, bipartite graphs, split graphs, very well covered graphs, chordal graphs and cactus graphs.

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# 1 Introduction

In this paper, we continue the study of domination in graphs. Let $G = (V, E)$ be a graph with vertex set $V$ and edge set $E$. A *dominating set* of $G$ is a set $D$ of vertices of $G$ such that every vertex in $V \setminus D$ is adjacent to a vertex in $D$. The *domination number* of $G$, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set. A dominating set of $G$ of cardinality $\gamma(G)$ is called a $\gamma(G)$-set. Domination in graphs is now well studied in graph theory. The literature on this subject has been surveyed and detailed in the two books by Haynes, Hedetniemi, and Slater [6, 7].

Let $G = (V, E)$ be a graph with no isolated vertex. A classical result in domination theory due to Ore [12] is that if $D$ is a minimum dominating set of $G$, then $V \setminus D$ is also a dominating set of $G$. A set $D'$ is an *inverse dominating set* of $G$ if $D'$ is a dominating set of $G$ and $D' \subseteq V \setminus D$ for some $\gamma(G)$-set $D$. By Ore's result, every graph with no isolated vertex has an inverse dominating set. The *inverse domination number* of $G$, denoted $\gamma^{-1}(G)$, is the minimum cardinality among all inverse dominating sets of $G$. An inverse dominating set of $G$ of cardinality $\gamma^{-1}(G)$ we call a $\gamma^{-1}(G)$-set. If $D'$ is a $\gamma^{-1}(G)$-set and $D$ is a $\gamma(G)$-set such that $D' \subseteq V \setminus D$, then we refer to the pair $(D, D')$ as an *inverse dominating pair*.

A set $I$ of vertices in $G$ is an *independent set* if no two vertices of $I$ are adjacent in $G$. The *independence number* of $G$, denoted $\alpha(G)$, is the maximum cardinality of an independent set of vertices in $G$. An *independent dominating set* of $G$ is a set that is both an independent set and a dominating set of $G$. The *independent domination number* of $G$, denoted by $i(G)$, is the minimum cardinality of an independent dominating set. By definition, $\gamma(G) \le i(G)$ for all graphs $G$.

Inverse domination in graphs was introduced by Kulli and Sigarkant [10]. In their original paper in 1991, they include a proof that for all graphs with no isolated vertex the inverse domination number is at most the independence number. However this proof is incorrect and contains an error. In 2004, Domke, Dunbar, and Markus [4] formally stated this "result" of Kulli and Sigarkant as a conjecture:

**Conjecture 1** (Domke, Dunbar, Markus [4]) *If $G$ is a graph with no isolated vertex, then $\gamma^{-1}(G) \le \alpha(G)$.*

For notation and graph theory terminology we in general follow [6]. Specifically, let $G = (V, E)$ be a graph with vertex set $V$ of order $n = |V|$ and edge set $E$ of size $m = |E|$, and let $v$ be a vertex in $V$. The *open neighborhood* of $v$ is the set $N(v) = \{u \in V \mid uv \in E\}$ and the *closed neighborhood of* $v$ is $N[v] = \{v\} \cup N(v)$. For a set $S \subseteq V$, its *open neighborhood* is the set $N(S) = \cup_{v\in S} N(v)$ and its *closed neighborhood* is the set $N[S] = N(S) \cup S$. A vertex $w \in V$ is an *$S$-private neighbor of* $v \in S$ if $N[w] \cap S = \{v\}$, while the *$S$-private neighbor set of* $v$, denoted $\mathrm{pn}(v, S)$, is the set of all $S$-private neighbors of $v$. Further if $w \in V \setminus S$, then $w$ is called an *external $S$-private neighbor of* $v$ and the *external $S$-private neighbor set of* $v$, denoted $\mathrm{epn}(v, S)$, is the set of all external $S$-private neighbors of $v$.

If $X, Y \subseteq V$, then the set $X$ is said to *dominate* the set $Y$ if $Y \subseteq N[X]$. In particular, if $X$ dominates $V$, then $X$ is a dominating set in $G$. If $X$ dominates $G$ and no subset of $X$ dominates $G$, then $X$ is called a minimal dominating set of $G$. The largest cardinality of a minimal dominating set for $G$ is the upper domination number of $G$, denoted $\Gamma(G)$.

For a set $S \subseteq V$, the subgraph induced by $S$ is denoted by $G[S]$. We denote the degree of $v$ in $G$ by $d_G(v)$, or simply by $d(v)$ if the graph $G$ is clear from context. The minimum degree (resp., maximum degree) among the vertices of $G$ is denoted by $\delta(G)$ (resp., $\Delta(G)$).

Two edges in a graph $G$ are *independent* if they are not adjacent in $G$. A set $M$ of pairwise independent edges of $G$ is called a *matching* in $G$ and if $2|M| = |V(G)|$ then $M$ is called a *perfect matching.*

If $G$ does not contain a graph $F$ as an induced subgraph, then we say that $G$ is $F$-free. In particular, we say a graph is *claw-free* if it is $K_{1,3}$-free.

A *block* of a graph $G$ is a maximal 2-connected subgraph of $G$. A graph $G$ is a *cactus* if and only if every block of $G$ is a cycle or a $K_2$. A graph $G$ is a *generalized cactus graph* (or a *generalized Gallai-tree*) if and only if every block of $G$ is a cycle or a complete graph. An *endblock* of $G$ is a block that contains only one cutvertex of $G$.

Our aim in this paper is to prove Conjecture 1 for some families of graphs. For this purpose, we recall that a *minimal dominating set* in $G$ is a dominating set that contains no dominating set as a proper subset. We shall need the following classical result due to Ore [12] of properties of minimal dominating sets.

**Lemma 1** (Ore [12]) *Let $D$ be a dominating set of a graph $G$. Then, $D$ is a minimal dominating set of $G$ if and only if for each $v \in D$, the vertex $v$ is isolated in $G[D]$ or $|epn(v, D)| \geq 1$.*

## 2 Special Families

In this section, we prove that Conjecture 1 is true for special families of graphs.

### 2.1 Claw-Free Graphs

First we establish that Conjecture 1 is true for claw-free graphs. For this purpose, we prove that every graph that has a minimum dominating set which is independent satisfies Conjecture 1.

**Theorem 1** *If $G$ is a graph with $\delta(G) \geq 1$ satisfying $\gamma(G) = i(G)$, then $\gamma^{-1}(G) \leq \alpha(G)$.*

**Proof.** Let $G = (V, E)$ and let $D$ be a $\gamma(G)$-set that is independent. Let $I$ be a maximal independent set of vertices in $G[V \setminus D]$. If $I = V \setminus D$, then $I$ is an inverse dominating set of $G$ that is independent, and so $\gamma^{-1}(G) \leq |I| \leq \alpha(G)$. Hence we may assume that $I \subset V \setminus D$. By the maximality of $I$, the set $I$ dominates $V \setminus D$. Let $A$ be the set of all vertices of $D$ not dominated by $I$ in $G$. If $A = \emptyset$, then $I$ is an inverse dominating set of $G$ that is independent, and so $\gamma^{-1}(G) \leq |I| \leq \alpha(G)$. Hence we may assume that $|A| \geq 1$. Since $D$ is an independent set and $\delta(G) \geq 1$, and since no vertex in $A$ is dominated by the set $I$, we note that $N(v) \subseteq V \setminus (D \cup I)$ for each $v \in A$. For each $v \in A$, let $v' \in N(v)$. Let $A' = \cup\{v'\}$ where the union is taken over all vertices $v \in A$. Then, $|A'| \leq |A|$ and $A' \cup I$ is an inverse dominating set of $G$. Hence, since $A \cup I$ is an independent set of $G$, we have that $\gamma^{-1}(G) \leq |A' \cup I| = |A'| + |I| \leq |A| + |I| = |A \cup I| \leq \alpha(G)$. $\Box$

Since every claw-free graph $G$ satisfies $\gamma(G) = i(G)$ ([1]), we have the following immediate consequence of Theorem 1.

**Corollary 1** *If $G$ is a claw-free graph with $\delta(G) \geq 1$, then $\gamma^{-1}(G) \leq \alpha(G)$.*

By applying Theorem 1 to graphs satisfying $\gamma(G) = \alpha(G)$ we get the following observation.

**Observation 1** *If a graph $G$ without isolated vertices satisfies $\gamma(G) = \alpha(G)$, then $G$ has two vertex-disjoint $\gamma(G)$-sets.*

From the proof of Theorem 1 it actually follows that in a graph $G$ without isolated vertices satisfying $\gamma(G) = \alpha(G)$, there is for each independent $\gamma(G)$-set $D$ a $\gamma(G)$-set contained in $V(G) \setminus D$.

### 2.2 Bipartite and Chordal Graphs

Since every minimal inverse dominating set in a graph without isolated vertices is a minimal dominating set in the graph, the inverse domination number is at most the upper domination number. Every maximal independent set in a graph is a minimal dominating set, and so the independence number of a graph is bounded above by its upper domination number. Hence we have the following observations.

**Observation 2** *If $G$ is a graph with $\delta(G) \geq 1$, then $\gamma^{-1}(G) \leq \Gamma(G)$ and $\alpha(G) \leq \Gamma(G)$.*

**Observation 3** *If $G$ is a graph with $\delta(G) \geq 1$ and $\alpha(G) = \Gamma(G)$, then $\gamma^{-1}(G) \leq \alpha(G)$.*

Several families of graphs $G$ are known to satisfy $\alpha(G) = \Gamma(G)$. These include, among others, bipartite graphs ([3]), chordal graphs ([9]), circular arc graphs ([5]), permutation graphs and comparability graphs ([2]). Hence, by Observation 3, Conjecture 1 is true for every graph in one of these families.

**Corollary 2** *If $G$ is a bipartite graph, a chordal graph, a circular arc graph, a permutation graph, or a comparability graph with $\delta(G) \geq 1$, then $\gamma^{-1}(G) \leq \alpha(G)$.*

### 2.3 Very Well-Covered Graphs

Recall that a graph $G$ of order $n$ is called *very well-covered* if $i(G) = \alpha(G) = n/2$. We show next that Conjecture 1 is true for every very well-covered graph. For this purpose, we first show that every graph that has a perfect matching and has independence number one-half its order satisfies Conjecture 1.

**Theorem 2** *If $G$ is a graph of order $n$ that has a perfect matching and $\alpha(G) = n/2$, then $\Gamma(G) = n/2$.*

**Proof.** Let $G = (V, E)$ and let $M$ be a perfect matching in $G$. By Observation 2, $\Gamma(G) \geq \alpha(G)$, and so $\Gamma(G) \geq n/2$. Let $D$ be a $\Gamma(G)$-set and let $V_i$ denote the vertices $v \in V$ such that $uv \in M$ and $|\{u, v\}| \cap D = i$ for $i \in \{0, 1, 2\}$. Then, $V = V_0 \cup V_1 \cup V_2$, $V_2 \subseteq D$ and $|D| = \frac{1}{2}|V_1| + |V_2|$. Since $D$ is a minimal dominating set of $G$, Lemma 1 implies that $|\text{epn}(x, D)| \geq 1$ for each vertex $x \in V_2$. Thus since each vertex of $V_1 \cap (V \setminus D)$ is dominated by $V_1 \cap D$, we must have that $\text{epn}(x, D) \subset V_0$ for each vertex $x \in V_2$. Hence,

$$|V_0| \geq |\bigcup_{x \in V_2} \text{epn}(x, D)| = \sum_{x \in V_2} |\text{epn}(x, D)| \geq |V_2|.$$

Therefore, $\Gamma(G) = |D| = \frac{1}{2}|V_1| + |V_2| \leq \frac{1}{2}(|V_0| + |V_1| + |V_2|) = n/2$. As observed earlier, $\Gamma(G) \geq n/2$. Consequently, $\Gamma(G) = n/2$. □

As an immediate consequence of Observation 3 and Theorem 2, we have the following result.

**Corollary 3** *If $G$ is a graph of order $n$ that has a perfect matching and $\alpha(G) = n/2$, then $\gamma^{-1}(G) \leq \alpha(G)$.*

Since every very well-covered graph has a perfect matching (e.g. see [13]) we obtain the following consequence of Corollary 3.

**Corollary 4** *If $G$ is a very well-covered graph of order $n$, then $\gamma^{-1}(G) \leq \alpha(G)$.*

## 2.4 Nearly Bipartite Graphs and Split Graphs

Recall that a graph $G$ is called a *split graph* if its vertices can be partitioned into two sets $X$ and $Y$ such that $X$ is an independent set and $G[Y]$ is a complete graph. As shown in Corollary 2, Conjecture 1 is true for the family of bipartite graphs. Here we establish that Conjecture 1 is true for the family of graphs that can be obtained from a bipartite graph by adding edges to one of its partite set such that each component of the subgraph induced by that set is a complete graph. As a special case of this result, we have that Conjecture 1 is true for split graphs.

**Theorem 3** *If $G$ is a graph without isolated vertices that can be obtained from a bipartite graph by adding edges to one of its partite set $Y$ such that each component of $G[Y]$ is a complete graph, then $\gamma^{-1}(G) \leq \alpha(G)$.*

**Proof.** Let $G = (V, E)$ be a graph without isolated vertices obtained from a bipartite graph $H$ with partite sets $X$ and $Y$ by adding edges to the subgraph $H[Y]$ induced by the partite set $Y$ in such a way that each component of $G[Y]$ is a complete graph.

Among all $\gamma(G)$-sets, let $D$ be one such that $|N(X \setminus D) \cap D|$ is a maximum. Let $D_X = D \cap X$ and $D_Y = D \cap Y$. Note that if $v \in D$ is not isolated in $G[D]$, then by Lemma 1, $|\text{epn}(v, D)| \geq 1$. Let $A = X \setminus D_X$ and let $B = N(A) \subseteq Y$. Hence, $B$ is the set of vertices in $Y$ dominated by the set $A$. Let $C$ be the set of vertices in $Y$ not dominated by $A$; that is, $C = Y \setminus B$. Let $I_C$ be a maximum independent set in $G[C]$ such that $|I_C \cap D|$ is a minimum.

We show that $I_C \cap D = \emptyset$. Assume, to the contrary, that there exists a vertex $v \in I_C \cap D$. Since $v \in C$, we have $\text{pn}(v, D) \subseteq Y$. Thus the vertex $v$ and the vertices from $\text{pn}(v, D)$ are in the same component $G_v$ of $G[Y]$. Let $C_v = V(G_v)$. For each vertex $y \in C_v \setminus \{v\}$, the set $D' = (D \setminus \{v\}) \cup \{y\}$ is a $\gamma(G)$-set since $G_v$ is a complete graph containing $\text{pn}(v, D)$. Further if $y \in B$, then $|N(A) \cap D'| > |N(A) \cap D|$, contradicting our choice of the set $D$. Hence, $C_v \subseteq C$. If $C_v = \{v\}$, then since $\delta(G) \geq 1$, $N(v) \subseteq D_X$ contradicting our earlier observation that $\text{pn}(v, D) \subseteq Y$. Hence, $D \cap C_v = \{v\}$ and $|C_v| \geq 2$. But then for each vertex $u \in C_v \setminus \{v\}$, $(I_C \setminus \{v\}) \cup \{u\}$ is a maximum independent set in $C$, contradicting our choice of the set $I_C$. Hence, $I_C \cap D = \emptyset$.

Let $Z$ be the set of vertices in $D_X$ that are not dominated by $I_C$ in $G$. Then, $A \cup I_C$ dominates $V \setminus Z$ and $A \cup I_C \cup Z$ is an independent set in $G$. For each $v \in Z$, let $v' \in N(v) \setminus D$ and let $Z' = \cup\{v'\}$ where the union is taken over all vertices $v \in Z$. Then, $|Z'| \leq |Z|$ and $A \cup I_C \cup Z'$ is an inverse dominating set of $G$. Hence, $\gamma^{-1}(G) \leq |A \cup I_C \cup Z'| = |A| + |I_C| + |Z'| \leq |A| + |I_C| + |Z| = |A \cup I_C \cup Z| \leq \alpha(G)$. $\square$

As an immediate consequence of Theorem 3, we have that Conjecture 1 is true for split graphs.

**Corollary 5** *If $G$ is a split graph with no isolated vertex, then $\gamma^{-1}(G) \leq \alpha(G)$.*

Note that since split graphs are chordal this result also follows from Corollary 2.

## 2.5 Cactus Graphs and Generalized Cactus Graphs

We prove in this section that if a graph only has cycles and complete graphs as blocks, then it satisfies Conjecture 1. It can easily be shown that if $G$ is a cactus such that each vertex contained in a $C_3$ is in no other cycle-block then $G$ can be obtained from a bipartite graph with no isolated vertex by adding the edges of a matching to one of its partite sets. Thus, by Theorem 3, such a graph $G$ satisfies Conjecture 1.

In the following we give an algorithm which for graphs only having cycles and complete graphs as blocks, can be used to construct an inverse dominating set with at most $\alpha(G)$ vertices. As in [6, 8] we label each vertex as Required, Bound or Free. Thus we partition $V(G)$ into three disjoint sets $R, B$ and $F$. Here $R$ denotes vertices required to be in our minimum dominating set under construction, $B$ denotes vertices not dominated by $R$, but bound to be dominated later on, and $F$, the free vertices, are vertices which need not to be dominated, either because they from the outset are declared to need no domination or because they already have been dominated in an earlier step of the algorithm. A vertex labelled $F$ may later have its label changed to $R$. The algorithm here starts with all vertices labelled $B$ and will stepwise change labelling until $R$ has grown into a minimum dominating set for $G$.

Mitchell, Cockayne, and Hedetniemi [11] presented a linear algorithm for finding a minimum $(R, B, F)$-dominating set $D$ for a rooted tree $T$ with root $x$ and with each vertex labelled by one of $R, B$ or $F$. That is, their algorithm finds a set $D$ of minimum cardinality such that $B \subseteq N[D]$. Further the algorithm constructs the set $D$ such that for each vertex $v$ the subtree containing $v$ and all its descendants contains as few vertices from $D$ as possible. Essentially the algorithm selects a dominating set by pushing $D$-vertices as far up the tree as possible. We shall use this tree algorithm given in [11] (it may also be found in [6]) to find a set $D$ of minimum cardinality such that $R \subseteq D$ and $B \subseteq N[D]$. Later we use the notation Tree$(T, x, R, B, F)$ for the vertices added to $R$ to obtain a $(R, B, F)$-dominating set by using this algorithm on the tree $T$ with root $x$ and where $(R, B, F)$ is a weak partition of $V(T)$. (By a *weak partition* of a set we mean a partition of the set in which some of the subsets may be empty.)

In the algorithm we construct an independent set $I$ and a set $S$ such that $S$ is an inverse dominating set. Further we define a function $s : S \to V(G)$ such that $s(v) \in N[v] \setminus R$ for each vertex $v \in S$.

To show correctness of the algorithm we use the following loop invariant, where $H$ is a subgraph of $G$ :

**Loop invariant :**

1. Each vertex in $R \cap V(H)$ is dominated by $S \setminus V(H)$.
2. $I$ is an independent set and $I \subseteq V(G) \setminus V(H)$.
3. If $x \in V(H) \setminus R$ and $N(x) \cap I \neq \emptyset$ then $x \in S$.
4. $S \subseteq V(G) \setminus R$ and for each set $A \subseteq S \cap V(H)$ the set $(S \setminus A) \cup (\bigcup_{v \in A} s(v))$ dominates $V(G) \setminus V(H)$ and $s(v) \in N[v] \setminus (R \cup V(H))$ for each $v \in S$.

5. $|S| \leq |I|$.

6. $R$ can be extended to a minimum dominating set of $G$ by adding vertices from $V(H)$.

7. $V(G) \setminus N[R] = B$.

In the algorithm we always consider an end-block $C$ of the current graph $H$ (or $C := H$ if $H$ does not have an end-block) with the three sets $R$, $B$ and $F$ given. If $x$ denotes the cut vertex in $C$ (or any vertex in $C$ if $H$ is a block), then we reduce the problem, such that we only have to consider the graph $H - (V(C) \setminus \{x\})$ (or the empty graph if $H \cong K_1$). In a step we may use auxiliary sets $D$ and $A$, $D$ to enlarge $R$ and $A$ to enlarge $I$ and $S$.

When adding vertices to $S$ and $I$ we have to make sure that 2), 3), and 6) are still satisfied. For this purpose we define the operation $extendIS(A, x, K)$ for an independent set of vertices $A$, the cut vertex $x$ of $C$ and a vertex-set $K \subseteq A$, where $|K| \leq 1$. This operation adds each vertex of $A \setminus N[I]$ both to $I$ and to $S$, and changes the function $s$ such that $s(v) = v$ for each $v \in A \setminus N[I]$. Further if $K = \{y\} \subseteq I$ (note that $I$ is changed now) and $x \notin R \cup S$ then the operation exchanges $y$ with $x$ in $S$, i.e., $S := (S \setminus \{y\}) \cup \{x\}$, sets $s(x) = y$ and cancels definition of $s(y)$.

If we add new vertices to the set $R$ that already belong to $S$ we have to change $S$ since $S$ should be a inverse dominating set and thus must have $S \cap R = \emptyset$. For this purpose the operation $addtoR(A)$ can be used if $A$ is the set of vertices added to $R$. The operation adds $A$ to the set $R$, changes $S$ to $(S \setminus A) \cup (\bigcup_{v \in A \cap S} s(v))$, $F$ to $F \cup (N(A) \cap B)$ and $B$ to $B \setminus (N[A] \cap B)$. Further, the definition of $s$ on vertices deleted from $S$ is cancelled and we set $s(v) = v$ for the new vertices just added to $S$.

**Algorithm : Inverse dominating set**
**Input** : A generalized cactus graph $G$

Let $R = F = \emptyset$, $B = V(G)$, $H = G$ and $I = S = \emptyset$.

While $H \neq \emptyset$ do

If $H$ contains a cut vertex, then let $C$ be a end-block in $H$; otherwise, let $C$ be the graph $H$. If $H \neq C$, then let $x$ be the unique cut vertex contained in $C$. If $H = C$, let $x$ be any vertex in $C$. Depending on $C$, $F$, $B$ and $R$ we now perform changes to $I, S, F, B$ and $R$ and after this we set $H := H - (V(C) \setminus \{x\})$ if $C \neq K_1$ and $H := \emptyset$ if $C = K_1$.

**Case 1:** $C \neq K_1$ is a complete graph.

If there exists a vertex $y \in (V(C) \setminus \{x\}) \cap B$, then perform the operations *addtoR*$(\{x\})$ and *extendIS*$(\{y\}, x, \emptyset)$. If there does not exist such a vertex and there is a vertex $y \in (V(C) \setminus \{x\}) \setminus N[S \setminus \{x\}]$, then perform the operation *extendIS*$(\{y\}, x, \{y\})$.

**Case 2:** $C \cong K_1$.

In this case, $H = C = K_1$. If $x \in B$, then perform the operation *addtoR*$(\{x\})$. If $N(x) \cap I = \emptyset$ then add $x$ to $I$ and let $S := S \cup \{v\}$ for a vertex $v \in N[x] \setminus R$ and set $s(v) := v$ (note that $N(x) \cap I \neq \emptyset$ implies $N[x] \cap S \neq \emptyset$).

**Case 3:** $C$ is a cycle.

**Case 3.1:** $V(C) \setminus N[x] \subseteq B$ and $N[x] \cap R = \emptyset$.

Assume $C\colon x, v_1, v_2, \ldots, v_{3k+i}, x$ where $i \in \{0, 1, 2\}$. Further assume, without loss of generality, that $v_1 \in F$ if $v_{3k+i} \in F$. If $v_1 \notin F$, then assume $v_1$ is adjacent to a vertex from $I$ if $v_{3k+i}$ is adjacent to a vertex from $I$. Depending on $F' := \{v_1, v_{3k+i}\} \cap F$ and $i$, let $D$ and $A$ be the sets from Table 1 and Table 2. Now perform the operations *addtoR*$(D)$ and *extendIS*$(A, x, A \cap \{v_{3k+i}\})$.

| $D$ | $F' = \emptyset$ | $F' = \{v_1\}$ | $F' = \{v_1, v_{3k+i}\}$ |
|---|---|---|---|
| $i = 0$ | $v_2, v_5, \ldots, v_{3k-1}$ | $v_3, v_6, \ldots, v_{3k}$ | $v_3, v_6, \ldots, v_{3k}$ |
| $i = 1$ | $x, v_3, v_6, \ldots, v_{3k}$ | $v_3, v_6, \ldots, v_{3k}$ | $v_3, v_6, \ldots, v_{3k}$ |
| $i = 2$ | $x, v_3, v_6, \ldots, v_{3k}$ | $x, v_3, \ldots, v_{3k}$ | $v_3, v_6, \ldots, v_{3k}$ |

Table 1: The set $D$ in Case 3.1

| $A$ | $F' = \emptyset$ | $F' = \{v_1\}$ | $F' = \{v_1, v_{3k+i}\}$ |
|---|---|---|---|
| $i = 0$ | $v_{3k}, v_1, v_4, \ldots, v_{3k-2}$ | $v_2, v_5, \ldots, v_{3k-1}$ | $v_2, v_5, \ldots, v_{3k-1}$ |
| $i = 1$ | $v_1, v_4, \ldots, v_{3k+1}$ | $v_{3k+1}, v_2, v_5, \ldots, v_{3k-1}$ | $v_2, v_5, \ldots, v_{3k-1}, v_{3k+1}$ |
| $i = 2$ | $v_1, v_4, \ldots, v_{3k+1}$ | $v_2, v_5, \ldots, v_{3k+2}$ | $v_2, v_5, \ldots, v_{3k-1}, v_{3k+1}$ |

Table 2: The set $A$ in Case 3.1

**Case 3.2 :** $V(C) \cap R \neq \emptyset$.

Let $e$ be an edge on $C$ incident with a vertex from $R$. Use the tree-algorithm to find $D := Tree(C - e, x, R \cap (V(C) \setminus \{x\}), B \cap (V(C) \setminus \{x\}), (F \cap V(C)) \cup \{x\})$. Thus the new $R$-dominators appointed are temporarily called $D$ until they are added to the set $R$. Now let $A'$ be the children, in the tree $C - e$ rooted at $x$, of the vertices from $D$ that are not contained in $R \cup D$ (note that by the properties of $D$ each vertex from $D$ has such a child). Let $A''$ be a maximal independent set in $C - (N[A'] \cup R \cup \{x\})$ constructed upwards from the bottom in $C - e$ (keep adding a vertex at maximal distance from $x$ and removing the closed neighborhood of that vertex) and let $A = A' \cup A''$. Now perform the operation *addtoR*$(D)$, and if $A$ contains a vertex $y \in N(x) \setminus N[I]$, then perform the operation *extendIS*$(A, x, \{y\})$; otherwise, perform the operation *extendIS*$(A, x, \emptyset)$.

**Case 3.3 :** There is a vertex $y \in (V(C) \cap F) \setminus N[x]$.

Using the tree-algorithm on $C - N[y]$ and $C - y$ it can be determined whether $\{y\} \cup R$ can be extended to a minimum dominating set or $R$ can be extended to a minimum dominating set not containing $y$.

If $\{y\} \cup R$ can be extended to a minimum dominating set, then let $D := \{y\} \cup Tree(C - y, x, (R \cap V(C)) \setminus (\{x\} \cup N[y]), (B \cap V(C)) \setminus (\{x\} \cup N[y]), (F \cap V(C)) \cup \{x\} \cup N[y])$. If $R$ can be extended to a minimum dominating set without $y$, then let $D := Tree(C - y, x, (R \cap V(C)) \setminus \{x\}, (B \cap V(C)) \setminus (\{x, y\}), (F \cap V(C)) \cup \{x, y\})$. Further, let $A'$ be the children, in $C - y$ rooted at $x$, of the vertices from $D \setminus \{y\}$ that are not contained in $R \cup D$. Let $A''$ be a maximal independent set in $C - (N[A'] \cup R \cup \{x, y\})$ constructed from the bottom up in $C - y$ and let $A = A' \cup A''$. Now perform the operation *addtoR*$(D)$.

First assume that one of three cases occur: (i) $y \in N(I)$, (ii) both vertices on $C$ adjacent to $y$ are in $A$ or (iii) exactly one of these neighbors $z$ is in $A$ but $A \cap (N(x) \setminus N[I]) \neq \{z\}$. If there is a vertex $u \in A \cap (N(x) \setminus N[I])$, then let $u$ be one with maximum distance to $y$ and perform the operation *extendIS*$(A, x, \{u\})$. Otherwise, perform the operation *extendIS*$(A, x, \emptyset)$.

Next it can be assumed that none of cases (i)-(iii) occur. Assume neither neighbor of $y$ is in $A$. If there is a vertex $z \in (N(x) \cap A) \setminus N(I)$, then perform the operation *extendIS*$(A, x, \{z\})$; otherwise, perform the operation *extendIS*$(A, x, \emptyset)$. Further, add the vertex $y$ to $I$ and a vertex $z \in N[y] \setminus R$ to $S$ and let $s(z) = z$. Otherwise it can be assumed that exactly one of the neighbors of $y$ is in $A$ and this neighbor $z$ satisfies $A \cap (N(x) \setminus N[I]) = \{z\}$. Now perform the operation *extendIS*$(A, x, \emptyset)$ and exchange the vertex $z$ with $y$ in $I$, setting $I := (I \setminus \{z\}) \cup \{y\}$.

**Output of algorithm : An inverse dominating set $S$ and an independent set $I$ of $G$ with $|S| \leq |I|$.**

**Theorem 4** *If $G \ncong K_1$ is a generalized cactus graph, then $\gamma^{-1}(G) \leq \alpha(G)$.*

**Proof.** First assume that the loop invariant is true for the loop in the algorithm. Since the loop invariant is obviously true when reaching the while-loop in the algorithm, the loop invariant is also true just before terminating the algorithm. By 6), $R$ is a $\gamma(G)$-set and by 4), $S \subseteq V(G) \setminus R$ and $S$ dominates $G$. Thus $S$ is an inverse dominating set, and by 2) and 5) the set $I$ is independent and $|S| \leq |I|$. It follows that $\gamma^{-1}(G) \leq |S| \leq |I| \leq \alpha(G)$.

Thus the theorem is true if the loop invariant can be verified. In all cases of the algorithm either we only add a single vertex $v \in V(C) \setminus \{x\}$ to $I$ not already adjacent to a vertex of $I$, or we add a subset of an independent set $A \subseteq V(C) \setminus \{x\}$ to $I$ by using the extend-operation ($extendIS(A, x, K)$). Here only the vertices from $A$ not adjacent to a vertex in $I$ are added to $I$ and thus the new set is independent and does not contain any vertices from the new graph $H$ resulting from this step after deletion of $C \setminus \{x\}$. Further, it follows that when using the extend-operation we add the same number of vertices to $I$ and $S$, and if vertices are added to $S$ but not by the extend-operation, then it is only a single vertex and at the same time a vertex is added to $I$. From these observations 2) and 5) remain satisfied.

Since vertices are only added to $R$ by using the operation $addtoR(D)$ for a set $D$ property 7) remains true since the add-operation changes $B$ to $B \setminus N[D]$ when adding $D$ to $R$.

When considering a block $C$ with the vertex $x$ in the algorithm it follows that if $D \subseteq V(C)$ is a set of minimum cardinality that contains a vertex as near to $x$ as possible such that $(V(C) \setminus \{x\}) \cap B \subseteq N[D]$, then $D \cup R$ can be extended to a minimum dominating set of $G$ by adding vertices from $V(H) \setminus V(C - x)$ if $R$ can be extended to a minimum dominating set by adding vertices from $V(H)$. In Case 1 and Case 3.1, the set $D$ is constructed such that it has this property, and in Case 3.2 and Case 3.3, the set $D$ gets this property since it is produced from the algorithm for trees. In Case 2 it follows that $R$ is a minimum dominating set if $x \notin B$; otherwise, $R \cup \{x\}$ is a minimum dominating set. Thus property 6) remains satisfied.

If a vertex $x \in R \cap V(H)$, then this vertex $x$ must have been added earlier when considering the block $C$ associated with the vertex $x$.

If the vertex $x$ is added to the set $R$ when considering a block, it can be seen that the extend-operation is used with a set containing a neighbor of $x$, and thus this neighbor is added to $S$ or is already in $S$. This is seen directly in Case 1 and Case 3.1. In Cases 3.2 and 3.3 it follows that $x \in D$ if $x$ is added to $R$ and in this case a neighbor $z$ of $x$ is in $A' \subseteq A$.

Since no vertex in $S \setminus V(H)$ is ever removed from $S$ property 1) remains true.

Further by looking at all cases it can be seen that if a neighbor of $x$ is added to $I$ when considering $C$, then either the add-operation is used with a set containing $x$ or the neighbor is added by using a extend-operation $extendIS(A, x, \{u\})$, where $u \in A$ is a vertex not adjacent to a vertex from $I$ and thus $x$ is added to $S$ except if $x$ is in $R$ or is already in $S$. Since a vertex is only removed from $S$ if it is added to $R$, property 3) remains true.

When proving 4), let $G_x$ be the component of $H - x$ containing $V(C - x)$. In the following we prove that after having considered $C$ and added vertices to $S$, the set $S \cap V(G_x)$ dominates $C - x$ if $x$ was not added to $S$, and if $x$ was added, then $S \cap (V(G_x)) \cup \{x\}$ and $S \cap (V(G_x)) \cup \{s(x)\}$ dominates $C - x$. By showing this, property 4) follows.

By considering Case 1, Case 2 and Case 3.1, this can easily be verified. In the other cases it can be shown in a similar manner and we only consider Case 3.3. By the properties of the domination-algorithm for trees it follows that the set $A'$ dominates $D$.

Since $A''$ is a maximal independent set in $C - (N[A'] \cup R \cup \{x, y\})$, the set $A = A' \cup A''$ dominates $V(C) \setminus (\{x, y\} \cup R)$, but all vertices from $R \cap (V(C - x))$ are already dominated from vertices from $(S \cap V(G_x)) \setminus V(C)$. Further by the choice of $A''$ it follows that if a neighbor $u$ of $x$ is in $A''$ then $(A \cup \{x\}) \setminus \{u\}$ also dominates $V(C) \setminus (\{x, y\} \cup R)$.

If $y \in N(I)$, both vertices on $C$ adjacent to $y$ are in $A$ or exactly one of these neighbors $z$ is in $A$ but $A \cap (N(x) \setminus N[I]) \neq \{z\}$ then the vertex $y$ will be dominated after using the extend operation. If this is not the case, a vertex from $N[y]$ is added to $S$ for the sole purpose to dominate $y$. Further all vertices from $C \setminus \{x, y\}$ are dominated since after the operation the set $S$ either contains $A$ or contains $A$ with a neighbor from $A''$ of $x$ exchanged with $x$. Thus property 4) follows in this case. □

# References


[1] R. H. Allan and R. C. Laskar, On domination and independent domination numbers of a graph. *Discrete Math.* **23** (1978), 73–76.

[2] G. A. Cheston and G. H. Fricke, Classes of graphs for which upper fractional domination equals independence, upper domination, and upper irredundance. *Discrete Appl. Math.* **27** (1990), 195–207.

[3] E. J. Cockayne, O. Favaron, C. Payan, and A. G. Thomason. Contributions to the theory of domination, independence and irredundance in graphs. *Discrete Math.* **33** (1981), 249–258.

[4] G. S. Domke, J. E. Dunbar and L. R. Markus, The inverse domination number of a graph. *Ars Combin.* **72** (2004), 149–160.

[5] M. C. Golumbic and R. C. Laskar. Irredundancy in circular arc graphs. *Discrete Appl. Math.* **44** (1993), 79–80.

[6] T. W. Haynes, S. T. Hedetniemi, and P. J. Slater, *Fundamentals of Domination in Graphs.* Marcel Dekker, New York, 1998.

[7] T. W. Haynes, S. T. Hedetniemi, and P. J. Slater (eds.), *Domination in Graphs: Advanced Topics.* Marcel Dekker, New York, 1998.

[8] S. T. Hedetniemi, R. Laskar and J. Pfaff, A linear algorithm for finding a minimum dominating set in a cactus. **13** (1986), 287-292.

[9] M. S. Jacobsen and K. Peters, Chordal graphs and upper irredundance, upper domination and independence. *Discrete Math.* **86** (1990), 59–69.

[10] V. R. Kulli and S. C. Sigarkant, Inverse domination in graphs. *Nat. Acad. Sci. Letters* **14** (1991), No. 12, 473–475.

[11] S. L. Mitchell, E. J. Cockayne, and S. T. Hedetniemi, Linear algorithms on recursive representations of trees. *J. Comput. System Sci.* **18** (1979), 76–85.

[12] O. Ore, *Theory of graphs. Amer. Math. Soc. Transl.* **38** (Amer. Math. Soc., Providence, RI, 1962), 206–212.

[13] B. Randerath and P. D. Vestergaard, Well-covered graphs and factors. *Disc. App. Math.* **154(9)** (2006), 1416–1428.

# Merrifield-Simmons Index and Minimum Number of Independent Sets in Short Trees

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**Abstract**

In Ars Comb. 84 (2007), 85–96, we posed the problem of determining a lower bound for the number of independent sets in a tree of fixed order and diameter $d$. Asymptotically, we give here a complete solution for trees of diameter $d \leq 5$. The lower bound is $5^{n/3}$ and we give the structure of the extremal trees. A generalization to connected graphs is stated.

**Keywords:** Independent sets, Merrifield-Simmons, Fibonacci number, diameter.
**AMS subject classification: 05C69, 05C05**

## 1 Introduction

Half a century ago authors counted maximal independent sets in a graph ([6], [7]) and the first results on the number of independent subsets of a graph appeared in [10], [2], [3], here $i(G)$ was called the Fibonacci number of $G$. In chemical literature $i(G)$ is called the Merrifield-Simmons index. It is treated in a monograph ([5]) and in a wealth of later papers ([1], [16], [15], [14], [13],[12], [11]).

In [9] we presented several upper and lower bounds for $i(G)$ in terms of order, size or independence number and obtained bounds for $i(G)$ on trees and unicyclic graphs.

If we denote $n$-order trees with diameter $d$ by $T(n, d)$, we have that

$$i(T(n,d)) \leq \operatorname{fib}(d) + 2^{n-d} \operatorname{fib}(d+1) \tag{1}$$

[8, Th. 3.1], [4, Th. 1].

Formula (1) gives a tight upper bound for the number of independent sets in a tree in terms of its diameter and order, in [8] we also determined the trees for which that upper bound is attained. In the same paper we posed the problem of determining the corresponding lower bound in terms of diameter and order, and asked for a characterization of the trees for which the lower bound is attained. This is for sufficiently large orders done here for diameters four and five. Asymptotically the number of independent sets in $n$-order trees of diameter five turns out to be $5^{n/3}$.

## 2 Notation

All graphs will be assumed simple and finite. Given a graph $G$, a subset $S$ of $V(G)$ is said to be independent, if no two vertices of $S$ are adjacent in $G$, in particular, the empty set is considered to be an independent set of any graph. The number of independent sets in a graph $G$ is denoted by $i(G)$.

We shall often consider some tree $T$ of a given diameter $d$ and order $n$ such that $i(T)$ is minimum. By this we mean that $T$ is a tree of diameter $d$ and order $n$ such that no other tree $T'$ of diameter $d$ and order $n$ contains fewer independent sets than $T$ does.

## 3 Helpful results

In this section we state some basic observations and results, which will be helpful for characterizing trees of a given diameter and order which contains the fewest possible number of independent sets.

**Observation 1.** Let $G$ be a graph and let $v \in V(G)$ and $e = uz \in E(G)$. Then
$(i)$ $\quad i(G) = i(G - v) + i(G - N[v])$
$(ii)$ $\quad i(G - e) = i(G) + i(G - N[\{u, z\}])$
$(iii)$ $\quad i(G) = i(G - \{u, z\}) + i(G - N[u]) + i(G - N[z])$

**Observation 2.** If $H$ is an induced subgraph of $G$ then $i(H) \leq i(G)$ and equality holds if and only if $G \cong H$. If $H$ is a spanning subgraph of $G$ then $i(H) \geq i(G)$ and equality holds if and only if $H \cong G$.

**Lemma 1.** Let $G$ be a graph with two leaves $l_1$ and $l_2$ such that $d(l_1, l_2) \leq 3$ and let $s_i$ denote the stem adjacent to $l_i$ for $i \in \{1, 2\}$. If $G' := G - s_2l_2 + l_1l_2$ then $i(G') \leq i(G)$ and if equality holds then either
(i) $d(l_1, l_2) = 2, s_1 = s_2$ and $N_G(s_1) = \{l_1, l_2\}$, i.e., in $G$ the three vertices $s_1, l_1, l_2$ span a $P_3$ as a component or
(ii) $d(l_1, l_2) = 3, s_1 \neq s_2$ and $N_G(s_2) = \{s_1, l_2\}$.

*Proof.* Observe that $G - l_2s_2 = G' - l_1l_2$ so by Observation 1(ii)

$$\begin{aligned} i(G) &= i(G') - i(G - N_G[\{l_2, s_2\}]) + i(G' - N_{G'}[\{l_1, l_2\}]) \qquad (2) \\ &= i(G') - i(G - N_G[s_2]) + i(G' - N_{G'}[l_1]). \end{aligned}$$

Since $G - N_G[s_2] \cong (G' - N_{G'}[l_1]) - N_{G'}(s_2)$ the graph $G - N_G[s_2]$ is an induced subgraph of $G' - N_{G'}[l_1]$. Therefore $i(G' - N_{G'}[l_1]) - i(G - N_G[s_2]) \geq 0$ and hence we have that $i(G) \geq i(G')$. If $i(G) = i(G')$ then $i(G' - N_{G'}[l_1]) - i(G - N_G[s_2]) = 0$ and for $d(l_1, l_2) = 2$, i.e., $s_1 = s_2$, we have $N_G(s_1) = \{l_1, l_2\}$ while for $d(l_1, l_2) = 3$, i.e., $s_1 \neq s_2$, we have that $N_G(s_2) = \{s_2, l_2\}$. This proves Lemma 1. $\square$

**Lemma 2.** Let $T$ be a tree of diameter $d \geq 4$ and order $n$ such that $i(T)$ is minimum. Then no vertex in $T$ is adjacent to more than two leaves, and if a vertex is adjacent to two leaves then it is penultimate on a diametrical path of $G$.

*Proof.* Assume that a vertex $v$ is adjacent to two leaves. By Lemma 1 it follows that $v$ is on a diametrical path $v_1, v_2 = v, v_3, v_4, \ldots, v_{d+1}$. So only the penultimate vertex of a diametrical path can support multiple leaves. We shall prove that $v$ can support at most two leaves. Let $L := \{l_1, \ldots, l_k\}$, $k \geq 3$, be the leaves adjacent to $v$ and consider the tree $T' := T - \{l_1v, l_2v\} + \{v_3l_1, l_1l_2\}$. Let $C$ be the component of $T - v_2v_3$ containing $v_3$ then

$$\begin{aligned} i(T) &= i(C - v_3)(2^{|L|} + 1) + i(C - N[v_3])2^{|L|} \\ &> 3i(C - v_3)(2^{|L|-2} + 1) + i(C - N[v_3])2^{|L|-1} = i(T'). \end{aligned}$$

Since $T'$ is a tree with diameter $d$ and order $n$, we have a contradiction with the minimality of $i(T)$. Thus, $v$ is not adjacent to more than two leaves. $\square$

**Lemma 3.** Let $H$ be a graph with a vertex $v$. Let $G_1, \ldots, G_k$, $k \geq 7$, be copies of $K_2$ and let $v_i \in G_i$. If $G = H \cup G_1 \cup \cdots \cup G_k + \{vv_1, \ldots, vv_k\}$ and $G' = H \cup G_1 \cup \cdots \cup G_{k-1} + \{x, y\} + \{vv_1, \ldots, vv_{k-1}, v_1x, v_2y\}$, then $i(G') < i(G)$.

*Proof.* By considering $G$ and $G'$ we observe that

$i(G) = 3^k i(H-v) + 2^k i(H-N[v])$ and $i(G') = 25 \cdot 3^{k-3} i(H-v) + 2^{k+1} i(H-N[v])$.

Thus, since $k \geq 7$, we obtain:

$i(G) - i(G') = 2 \cdot 3^{k-3} i(H-v) - 2^k i(H-N[v]) \geq (2 \cdot 3^{k-3} - 2^k) i(H-N[v]) > 0.$

$\square$

## 4 Trees of diameter three

For trees of diameter three the problem is straightforward. For completeness we describe the trees $T$ of diameter three for which $i(T)$ is minimum.

**Proposition 1.** Given any fixed $n \geq 4$, let $T$ denote a tree of diameter three and order $n$ for which the number of independent sets is minimum. Let $P_4 : x_0x_1x_2x_3$ denote a diametrical path of $T$. Then

$$\{\deg(x_1), \deg(x_2)\} = \left\{ \left\lfloor \frac{n-2}{2} \right\rfloor, \left\lceil \frac{n-2}{2} \right\rceil \right\}.$$

## 5 Trees of diameter four

Let $G_{2k+2}$, $k \geq 2$, be the graph obtained from $K_{1,k+1}$ by subdividing $k$ of its edges. Consider a tree $T$ with diameter 4 and order $n$ such that $i(T)$ is minimum. Let $v_1, \ldots, v_5$ be a diametrical path in $T$. If $n \geq 7$ it follows from Lemma 1 and Lemma 2 that $T \cong G_n$ or that each component of $T - v_3$ is isomorphic to $K_2$ or $P_3$. If $T \ncong G_n$ then let $s(T)$ and $t(T)$ denote the number of components from $T - v_3$ isomorphic to $K_2$ and $P_3$, respectively. Then $n = 1 + 2s(T) + 3t(T)$ and $i(T) = 2^{s(T)}4^{t(T)} + 3^{s(T)}5^{t(T)}$.

**Theorem 1.** Let $T_n$ be a tree of diameter four and order $n$ for which the parameter $i(T_n)$ attains its minimum value and let $v_1, \ldots, v_5$ be a diametrical path in $T_n$. Then $T_5 = P_5$, $T_6 = G_6$ and if $n \geq 7$ then each component of $T_n - v_3$ is isomorphic to $K_2$ or $P_3$ and

- $s(T_n)$ is as indicated in the following table when $7 \leq n \leq 25$.
- $s(T_n) = 2n + 1 \; mod \; 3$ for $n \geq 26$.

| n | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|---|---|---|---|---|---|---|---|---|---|---|
| s(T) | 3 | 2 | 4 | 3 | 5 | 4 | 6 | 5 | 4 | 3 |

| n | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 |
|---|---|---|---|---|---|---|---|---|---|
| s(T) | 5 | 4 | 3 | 2 | 4 | 3 | 2 | 1 | 3 |

*Proof.* The theorem is easily verified for $n \leq 6$. Thus, we may assume $n \geq 7$. By considering $G_n$ (if $n$ is even) it easily follows that the graph $T'$ obtained from $G_n$ by removing the leaf adjacent to the center vertex and attaching a second leaf to another stem satisfies $i(T') < i(G_n)$. Thus we may assume that $T_n \ncong G_n$ and only $s(T_n)$ has to be determined.

Now consider trees $T'$ and $T''$ with the same structure as $T$ such that $s(T'') = s(T') - 3 \geq 0$ and $t(T'') = t(T') + 2$. If $s' := s(T')$ and $t' := t(T'')$ then

$$i(T') - i(T'') = \frac{2}{27} 3^{s'} 5^{t'} - 2^{s'+2t'}.$$

It follows that

$$i(T') - i(T'') \geq 0 \Leftrightarrow \left(\frac{3}{2}\right)^{s'} \left(\frac{5}{4}\right)^{t'} \geq \frac{27}{2} \Leftrightarrow s' \log 3/2 + t' \log 5/4 \geq \log 27/2.$$

Since $n' := |V(T')| = 1 + 2s' + 3t'$ we may obtain that $i(T') - i(T'') \geq 0$ if and only if $s' \geq a - bn'$ for real numbers $a$ and $b$ ($a \approx 10,429$ and $b \approx 0,2898$).

It follows that if $k$ is the largest integer such that $k \leq a - bn$ and $n = 1 + 2k + 3t$ for some integer $t \geq 0$ then $s(T_n) = k$ if and only if $k \geq 3$. Using these observations, it is straightforward to derive the values of $s(T_n)$ for $n \leq 25$. For $n \geq 26$ the inequality $s' \leq a - bn$ implies that $s' < 3$ and therefore $s(T_n) \leq 2$. By the equation $n = 1 + 2s(T) + 3t(T)$ we obtain that $2s(T_n) \equiv n - 1 (\text{mod } 3)$ and the statement is obtained since this implies that $s(T_n) \equiv -2s(T_n) \equiv 1 - n \equiv 2n + 1 (\text{mod } 3)$. □

## 6 Trees of diameter five

In order to describe trees of diameter five with minimum number of independent sets we introduce the following terminology.

Let $T$ denote a tree of diameter five with a diametrical path $P_6 : x_0x_1x_2x_3x_4x_5$. If there is exactly one leaf attached to $\{x_2, x_3\}$, then we refer to $T$ as a *center-leaf tree*, and if there is no leaf attached to $\{x_2, x_3\}$, then we refer to $T$ as a *center-leaf-free tree*.

Let $T$ denote a center-leaf-free tree. If every component of $T - \{x_2, x_3\}$ is a $K_{1,1}$ then $T$ is referred to as a center-leaf-free $K_{1,1}$-tree. If every component of $T - \{x_2, x_3\}$ is a $K_{1,2}$, then $T$ is referred to as a center-leaf-free $K_{1,2}$-tree. If every component of $T - \{x_2, x_3\}$ is a $K_{1,1}$ or a $K_{1,2}$, then $T$ is referred to as a center-leaf-free mixed-$K_{1,1}$-$K_{1,2}$-tree.

## 6.1 Some lemmas concerning trees of diameter five

In the following we prove some results needed for the characterization of trees of diameter five with minimum number of independent sets.

**Lemma 4.** Let $T$ be a tree of diameter five for which $i(T)$ is minimum, and let $P_6$ : $x_0x_1x_2x_3x_4x_5$ denote a diametrical path of $T$. Then

(1) The neighbourhood $N[x_2, x_3]$ contains at most one leaf,

(2) if there is a leaf $l$ attached to either $x_2$ or $x_3$, then every component of $T - \{x_2, x_3, l\}$ is a $K_{1,1}$, and

(3) if neither $x_2$ nor $x_3$ has a leaf attached, then every component of $T - \{x_2, x_3\}$ is a $K_{1,1}$ or a $K_{1,2}$.

*Proof.* Statement (1) follows from Lemma 1, while statement (3) follows from Lemma 2. To prove statement (2), we may assume that a leaf $l$ is adjacent to $x_2$. From Lemma 1 it follows that all vertices from $N(x_2)\backslash\{x_3\}$ have degree at most two in $T$. Thus we may assume that a vertex $y \in N(x_3)\backslash\{x_2\}$ has degree at least three in $T$.

Let $l'$ be a leaf adjacent to $y$ and consider the tree $T' := T - yl' + ll'$. Observe that an independent set $S$ in $T'$ is independent in $T$ unless $\{l', y\} \subseteq S$. An independent set in $T$ containing both $l$ and $l'$ is not independent in $T'$. Therefore

$$i(T) = i(T') - i(T' - N_{T'}[l', y]) + i(T - N_T[l, l']). \qquad (3)$$

Since $T' - N_{T'}[l', y]$ is an induced subgraph of a graph that has $T - N_T[\{l, l'\}]$ as a spanning subgraph and $T' - N_{T'}[\{l', y\}] \ncong T - N_T[\{l, l'\}]$, we have by Observation 2 that $i(T' - N_{T'}[l', y]) < i(T - N_T[l, l'])$. Thus (3) implies $i(T) > i(T')$ which contradicts the choice of $T$. $\square$

Lemma 4 states that given an integer $n \geq 6$, the minimum value of $i(T)$ over all trees of order $n$ and diameter five is attained for a center-leaf $K_{1,1}$-tree or a center-leaf-free mixed-$K_{1,1}$-$K_{1,2}$-tree. Given any center-leaf-free mixed-$K_{1,1}$-$K_{1,2}$-tree $T$, we let $p(T)$ and $q(T)$ denote the number of $K_{1,1}$'s attached to $x_2$ and $x_3$, respectively, and let $r(T)$ and $s(T)$ denote the number of $K_{1,2}$'s attached, by their center vertex, to $x_2$ and $x_3$, respectively. Whenever the context is clear we will simply write $p, q, r$ and $s$ for $p(T), q(T), r(T)$ and $s(T)$. For the number of independent sets in a center-leaf-free mixed-$K_{1,1}$-$K_{1,2}$-tree $T$ we can use Observation 1(iii) to obtain Proposition 2 below.

**Proposition 2.** The number of independent sets in any center-leaf-free mixed-$K_{1,1}$-$K_{1,2}$-tree $T$ is

$$3^{p+q}5^{r+s} + 2^p 3^q 4^r 5^s + 2^q 3^p 5^r 4^s. \tag{4}$$

**Lemma 5.** Let $T$ be a tree of diameter five for which $i(T)$ is minimum. If there is a leaf attached to $x_2$ or $x_3$, then $n(T) \leq 27$.

*Proof.* Suppose there is a leaf attached to $x_2$ or $x_3$. Then it follows from Lemma 4, that $T$ is a center-leaf $K_{1,1}$-tree. If $p$ and $q$ denote the number of $K_2$'s attached to $x_2$ and $x_3$, respectively, then $p, q \leq 6$, according to Lemma 3. Since $n(T) = 3 + 2(p+q)$, the desired bound on $n(T)$ follows. □

**Corollary 1.** If $n \geq 28$, then a tree $T$ of diameter five and order $n$ for which $i(T)$ is minimum is a center-leaf-free mixed-$K_{1,1}$-$K_{1,2}$-tree.

*Proof.* As noted above in Lemma 4, the tree $T$ is either a center-leaf $K_{1,1}$-tree or a center-leaf-free mixed-$K_{1,1}$-$K_{1,2}$-tree. Since $n \geq 28$, the claim follows from Lemma 5. □

**Lemma 6.** Let $T$ be a tree of diameter five for which $i(T)$ is minimum, and let $P_6 : x_0x_1x_2x_3x_4x_5$ denote a diametrical path of $T$. Let $r$ and $s$ denote the number of $K_{1,2}$'s attached by their center vertex to $x_2$ and $x_3$, respectively. By symmetry, we may assume $r = s + c$ for some non-negative integer $c$. If $c \geq 1$ then $p \leq q$ and given values of $p$ and $q$ we have that $c$ is the largest possible integer such that

$$c \leq \left\lfloor \frac{\log(5/4) + (q-p)\log(3/2)}{\log(5/4)} \right\rfloor.$$

*Proof.* Suppose that $c \geq 1$. Let $T'$ denote the center-leaf-free mixed $K_{1,1}$-$K_{1,2}$-tree with $p(T') = p$, $q(T') = q$, $r(T') = r - 1 = s + c - 1$ and $s(T') = s + 1$. According to Observation 1(iii),

$$\begin{aligned} i(T) &= 3^{p+q}5^{r+s} + 2^p 3^q 4^r 5^s + 2^q 3^p 5^r 4^s \quad \text{and} \\ i(T') &= 3^{p+q}5^{r+s} + 2^p 3^q 4^{r-1} 5^{s+1} + 2^q 3^p 5^{r-1} 4^{s+1}. \end{aligned}$$

Thus $i(T) - i(T') = \frac{1}{5}2^q 3^p 5^r 4^s - \frac{1}{4}2^p 3^q 4^r 5^s$. Now consider the logarithm of the ration of these two terms

$$\log\left(\frac{\frac{1}{5}2^q 3^p 5^r 4^s}{\frac{1}{4}2^p 3^q 4^r 5^s}\right) = \log\left(\frac{4}{5}\left(\frac{2}{3}\right)^{q-p}\left(\frac{5}{4}\right)^c\right) = \log\left(\frac{4}{5}\right) + (q-p)\log\left(\frac{2}{3}\right) + c\log\left(\frac{5}{4}\right).$$

This term is at most zero since $i(T) \leq i(T')$. This implies that $p \leq q$ and $c$ must be the largest possible integer such that

$$c \leq \frac{\log(5/4) + (q-p)\log(3/2)}{\log(5/4)}.$$

$\square$

Analogously to Lemma 3 we can obtain Lemma 7.

**Lemma 7.** Let $H$ be a graph with a vertex $v$. Let $G_1, \ldots, G_9$ be copies of $K_{1,2}$ and let $v_i$ be the center vertex of $G_i$, $1 \leq i \leq 9$. Let $F_1, F_2, F_3$ be copies of $K_{1,1}$ and let $f_i$ be a vertex in $F_i$, $1 \leq i \leq 3$. If $G = H \cup G_1 \cup \cdots \cup G_7 \cup F_1 \cup F_2 \cup F_3 + \{vv_1, \ldots, vv_7, vf_1, vf_2, vf_3\}$ and $G' = H \cup G_1 \cup \cdots \cup G_9 + \{vv_1, \ldots, vv_7, vv_8, vv_9\}$, then $i(G') < i(G)$.

*Proof.* $i(G) - i(G') = 2 \cdot 5^7 \cdot i(H-v) - 2^{17} i(H - N[v]) > 0$ because $i(H-v) \geq i(H-N[v])$ and $5^7 > 2^{16}$. $\square$

From Lemma 7 we obtain the following corollary.

**Corollary 2.** Let $T$ be a tree of diameter five for which $i(T)$ is minimum. If $n \geq 88$ then $p \leq 2, q \leq 2$ and $|r-s| \leq 4$.

*Proof.* Assume that $p \geq 3$ or $q \geq 3$. Since $p \leq 6, q \leq 6$ and $|r-s| \leq 11$ by Lemma 6 the assumption $n \geq 88$ implies that either $p \geq 3$ and $r \geq 7$ or $q \geq 3$ and $s \geq 7$. Now Lemma 7 implies that $i(T)$ is not minimum.

By using Lemma 6 and $p \leq 2$, $q \leq 2$ we obtain that $|r-s| \leq 4$ and if $r > s$ then $p \leq q$. $\square$

## 6.2 Main result for trees of diameter five

By using the results from Section 6.1 we obtain the main results for trees of diameter five.

**Theorem 2.** For any $n \geq 28$, a tree $T$ of diameter five and order $n$ for which $i(T)$ is minimum is a center-leaf-free mixed-$K_{1,1}$-$K_{1,2}$-tree with $r(T) = s(T) + c$ and $q(T) = p(T) + d$ for non-negative integers $c$ and $d$. Moreover, $c \leq 11$ and $p(T), q(T) \leq 6$. If $n \geq 88$ then $c \leq 4$ and $p(T), q(T) \leq 2$.

*Proof.* The proof relies on the results of Section 6.1. According to Corollary 1, the tree $T$ as described in the theorem is a center-leaf-free mixed-$K_{1,1}$-$K_{1,2}$-tree. The bounds on the parameters $p(T), q(T), r(T), s(T)$ follows from Lemma 3, Lemma 6, and Corollary 2. $\square$

If $T_n$ is a tree of diameter five and order $n$ for which $i(T_n)$ is minimum, then it follows from the above theorem that as $n$ increases the tree $T_n$ will be an increasingly 'well-balanced' center-leaf-free mixed-$K_{1,1}$-$K_{1,2}$-tree, that is, the ratio of $r(T_n)$ and $s(T_n)$ will tend to one, and the ratio of $(p(T_n) + q(T_n))/n$ will be small.

**Lemma 8.** There is an integer $n'$ such that if $T_n$ is a tree of diameter five and order $n \geq n'$ for which $i(T_n)$ is minimum then $p(T_n) + q(T_n) \leq 2$.

*Proof.* Let $T^n$ be any center-leaf-free mixed-$K_{1,1}$-$K_{1,2}$-tree of order $n$ such that $p(T^n)+q(T^n) \geq 3$ and $p(T^n), q(T^n) \leq 6$. Consider a center-leaf-free mixed-$K_{1,1}$-$K_{1,2}$-tree $T_2^n$ of order $n$ with $p(T_2^n)+q(T_2^n) = p(T^n)+q(T^n)-3$ and $r(T_2^n)+s(T_2^n) = r(T^n)+s(T^n)+2$. From equation (4) it follows that

$$\lim_{n\to\infty} \frac{i(T^n)}{3^{p(T^n)+q(T^n)}5^{r(T^n)+s(T^n)}} = 1 \quad \text{and} \quad \lim_{n\to\infty} \frac{i(T_2^n)}{3^{p(T^n)+q(T^n)}5^{r(T^n)+s(T^n)}} = \frac{25}{27}.$$

Thus there must exist an integer $n'$ such that $i(T^n) > i(T_2^n)$ when $n \geq n'$.This implies that $T_n$ can not be the graph $T^n$ for $n \geq n'$ and the statement follows. □

By using the result from Lemma 8 we can obtain the following characterization of $T_n$ when $n \geq n'$.

**Theorem 3.** There is an integer $n'$ such that if $T_n$ is a tree of diameter five and order $n \geq n'$ for which $i(T_n)$ is minimum. Then $T_n$ is a center-leaf-free mixed-$K_{1,1}$-$K_{1,2}$-tree and $p, q, r$ and $s$ is as indicated in the following table (it is assumed that $c := r - s \geq 0$ and $p \leq q$):

| n mod 6 | q | p | c |
|---|---|---|---|
| 0 | 1 | 1 | 0 |
| 1 | 1 | 0 | 1 |
| 2 | 0 | 0 | 0 |
| 3 | 2 | 0 | 3 |
| 4 | 1 | 0 | 2 |
| 5 | 0 | 0 | 1 |

*Proof.* Let $n'$ be the integer from Lemma 8 and consider $T_n$ when $n \geq n'$. Then $p+q \leq 2$ and Lemma 6 implies that $c \leq 1$ if $p = q$, $c \in \{1, 2\}$ if $q-p = 1$ and $c \in \{3, 4\}$ if $q-p = 2$. By considering cases depending on $n \mod 6$ it can be observed that this determine the parameters $p, q$ and $c$ when $n \mod 3 \neq 0$. Further in the case $n \mod 6 = 0$ either $p = 1, q = 1$ and $c = 0$ or $p = 0, q = 2$ and $c = 4$ and in the case $n \mod 6 = 3$ either $p = 1, q = 1$ and $c = 1$ or $p = 0, q = 2$ and $c = 3$. In both cases we only have to compare the number of independent sets in the two trees that might be isomorphic to $T_n$ and the result is as indicated in the table. □

It can be shown that the integer $n'$ form Lemma 8 and Theorem 3 can be choosen to be smaller that one hundred.

From Theorem 3 we immediately obtain

**Corollary 3.** Asymptotically the minimum number of independent sets in $n$-order trees of diameter five is $5^{n/3}$.

# 7 The lower bound of $i$ on graphs of fixed order and diameter

The following theorem gives an optimal bound for $i$ for connected graphs of fixed order and diameter. The graph obtained by attaching a path $P$ to a vertex $v$ in a graph $G$ is the graph $P \cup G + uv$ where $u$ is a vertex from $P$ with minimum degree. The Fibonacci numbers $fib(0), fib(1), \ldots$ are defined by the equations $fib(0) := 0$, $fib(1) := 1$ and $fib(n) := fib(n-1) + fib(n-2)$ for $n \geq 2$.

**Theorem 4.** If $G$ is a connected graph of order $n$ and diameter $d \geq 2$, then

$$2fib(d+1) + (n-d)fib(d) \leq i(G), \tag{5}$$

where equality occurs if and only if $G$ is isomorphic to the graph obtained from $K_{n-d+2}$ by removing an edge $wv$ and attaching a path $P_{d-2}$ at $v$ (if $d \geq 3$).

*Proof.* If $G \cong P_{d+1}$ then the statement is true for $G$ since $i(G) = fib(d+3) = 2fib(d+1) + (n-d)fib(d)$. Let $G$ be a connected graph of order $n$ and diameter $d$, $G \ncong P_{d+1}$. Assume that the statement is true for each graph of order less than $n$. Consider a diametrical path $P : v_1, \ldots, v_{d+1}$ in $G$. Since $G \ncong P_{d+1}$ there must be a vertex $u \notin V(P)$ such that $G-u$ is connected and since $P$ is a diametrical path $u$ can at most be adjacent to three vertices from $P$. Thus $G-u$ is a graph with diameter at least $d$ and $G - N[u]$ has at least $d-2$ vertices. By assumption we have that $i(G-u) \geq 2fib(d+1) + (n-1-d)fib(d)$ and Observation 2 implies that $i(G-N[u]) \geq i(P_{d-2}) = fib(d)$. If equality holds in both inequalities then $G - N[u] \cong P_{d-2}$ and $G-u$ has diameter $d$ and can be constructed as one of the graphs described in the statement. Thus if equality holds in both inequalities $G$ must be one of the graphs described in the statement. By applying Observation 1 we obtain that

$$i(G) = i(G-u) + i(G-N[u]) \geq 2fib(d+1) + (n-d)fib(d)$$

and equality occurs if and only if $G$ is one of the graphs described in the statement. □

# References

[1] Ivan Gutman. Extremal hexagonal chains. *J. Math. Chem.*, 12(1-4):197–210, 1993. ISSN 0259-9791. Applied graph theory and discrete mathematics in chemistry (Saskatoon, SK, 1991).

[2] Peter Kirschenhofer, Helmut Prodinger, and Robert F. Tichy. Fibonacci numbers of graphs. II. *Fibonacci Quart.*, 21(3):219–229, 1983. ISSN 0015-0517.

[3] Peter Kirschenhofer, Helmut Prodinger, and Robert F. Tichy. Fibonacci numbers of graphs. III. Planted plane trees. In *Fibonacci numbers and their applications (Patras, 1984)*, volume 28 of *Math. Appl.*, pages 105–120. Reidel, Dordrecht, 1986.

[4] Xueliang Li, Haixing Zhao, and Ivan Gutman. On the Merrifield-Simmons index of trees. *MATCH Commun. Math. Comput. Chem.*, 54(2):389–402, 2005. ISSN 0340-6253.

[5] Richard E. Merrifield and Howard E. Simmons. *Topological Methods in Chemistry.* Wiley-Interscience, 1989.

[6] R. E. Miller and D.E. Muller. A problem of maximum consistent subsets. Technical report, IBM research report RC-240, J.T. Watson Research Center, 1960.

[7] J. W. Moon and L. Moser. On cliques in graphs. *Israel J. Math.*, 3:23–28, 1965. ISSN 0021-2172.

[8] Anders Sune Pedersen and Preben Dahl Vestergaard. An upper bound on the number of independent sets in a tree. *Ars Comb.*, 84:85–96, 2007.

[9] Anders Sune Pedersen and Preben Dahl Vestergaard. Bounds on the number of vertex independent sets in a graph. *Taiwanese J. Math.*, 10(6):1575–1587, 2006. ISSN 1027-5487.

[10] Helmut Prodinger and Robert F. Tichy. Fibonacci numbers of graphs. *Fibonacci Quart.*, 20(1):16–21, 1982. ISSN 0015-0517.

[11] H. Wang and H. Hua. Unicycle graphs with extremal merrifield-simmons index. *J. of Math. Chem.*, 43(1):202–209, 2008.

[12] M. Wang, H. Hua, and D: Wang. The first and second largest merrifield-simmons indices of trees with prescribed pendent vertices. *Journal of Mathematical Chemistry*, 43(2):727–736, 2008.

[13] A. Yu and X. Lv. The merrifield-simmons indices and hosoya indices of trees with $k$ pendent vertices. *J. Math. Chem.*, 41(1), 2007.

[14] Aimei Yu and Feng Tian. A kind of graphs with minimal Hosoya indices and maximal Merrifield-Simmons indices. *MATCH Commun. Math. Comput. Chem.*, 55(1):103–118, 2006. ISSN 0340-6253.

[15] Lian-Zhu Zhang and Feng Tian. Extremal catacondensed benzenoids. *J. Math. Chem.*, 34(1-2):111–122, 2003. ISSN 0259-9791.

[16] Lianzhu Zhang. The proof of gutman's conjectures concerning extremal hexagonal chains. *Journal of Systems Science and Mathematical Science*, 18(4):460–465, 1998.

# The Number of Independent Sets in Graphs with Minimum Degree 2 or 3 or Graphs with $m$ Edges

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**Abstract**

Since its start in the 1960's there has been some interest in counting independent sets in a graph. At first the main interest were the maximal independent sets and maximum independent sets of graphs but later all independent sets in graphs have been considered. For maximal independent sets and maximum independent sets status reports are given in the articles "Survey on counting maximal independent sets" [1] and "The number of maximum independent sets in graphs" [2]. Here we present three new results for $i(G)$, the number of independent sets in a graph $G$. The first result proves that an unicyclic graph $G$ with $n$ vertices and girth $k$ satisfies that $i(G) \geq fib(n+1) + fib(k-1)fib(n-k+2)$. Secondly, we prove that a connected graph of order $n$ and size $n-1+k$ satisfies $i(G) \leq 1 + 2^{n-k-2} + 2^{n-2}$ and if equality occurs then $G$ is the complete bipartite graph $K_{k,n-k}$ or $G$ is a star $K_{1,n-1}$ with $k$ edges added. Finally it is proven for $k \in \{2,3\}$ that a connected graph of order $n$ with $\delta(G) \geq k$ satisfies $i(G) \leq 2^{n-k} + 2^k - 1$ with equality if and only if $G \cong C_5$ or $G \cong K_{k,n-k}$.

**Keywords:** Independent sets, Merrifield-Simmons, Fibonacci number, unicyclic graphs, minimum degree.

**AMS subject classification: 05C69, 05C05**

## 1 Introduction and Preliminary Results

For a graph $G$ a subset $S \subseteq V(G)$ is said to be independent if no two vertices in $S$ are adjacent. The number of independent sets in a graph $G$ is denoted by $i(G)$. For $v \in V(G)$ the set of vertices adjacent to $v$ is denoted by $N(v)$ and we define $N[v] := N(v) \cup \{v\}$. Further we define $N(S)$ and $N[S]$ for a vertex set

$S \subseteq V(G)$ by

$$N(S) := \bigcup_{v \in S} N(v) \qquad N[S] := \bigcup_{v \in S} N[v].$$

Below we state some preliminary results that will be used later.

**Observation 1.**
Let $G$ be a graph and let $G_1, \ldots, G_k$ denote the components of $G$ then

$$i(G) = \prod_{i=1}^{k} i(G_i).$$

**Lemma 1.**
If $v$ is a vertex in a graph $G$ then $i(G) = i(G - v) + i(G - N[v])$.

**Theorem 1.**
If $G_i$ is an induced subgraph of $G \ncong G_i$ then $i(G) > i(G_i)$. If $G_u$ is a spanning subgraph of $G \ncong G_u$ then $i(G_u) > i(G)$.

The Fibonacci numbers are defined recursively by $fib(0) = 0$, $fib(1) = 1$, and

$$fib(n) = fib(n-2) + fib(n-1) \quad \text{for} \quad n \geq 2.$$

We have the following result for the path $P_n$ and the circuit $C_n$ of order $n$.

**Theorem 2.**
$i(P_n) = fib(n+2)$ for $n \geq 0$ and $i(C_n) = fib(n-1) + fib(n+1)$ for $n \geq 3$.

In [5] the following result was proven.

**Theorem 3.**
Let $T$ be a tree of order $n$ then $fib(n+2) \leq i(T) \leq 2^{n-1} + 1$. Further $i(T) = fib(n+2)$ if and only if $T \cong P_n$, and $i(T) = 2^{n-1} + 1$ if and only if $T \cong K_{1,n-1}$ or $T \cong K_1$.

In [3] the following result was proven.

**Theorem 4.**
Let $F$ be a forest of order $n \geq 6$ such that $F \ncong P_n$, then

$$i(F) \geq 2fib(n) + 3fib(n-3).$$

## 2 Independent sets in unicyclic graphs with a given girth

In the following we shall give a lower bound for the number of independent sets in a unicyclic graph expressed by its order and girth. First we define a lollipop $L_{n,k}$.

**Definition 1.**
For integers $n$ and $k$, where $3 \leq k \leq n$, we let $L_{n,k} \cong C_k$ if $n = k$ and otherwise $L_{n,k}$ is defined as the graph $C_k \cup P_{n-k} + xy$ where $x \in V(C_k)$ and $y$ is a vertex in $P_{n-k}$ such that $deg_{P_{n-k}}(y) = \delta(P_{n-k})$.

The lower bound for the number of independent sets in unicyclic graphs can now be proven.

**Theorem 5.**
Let $G$ be an unicyclic graph of order $n$ and with girth $k$, then

$$i(G) \geq fib(n+1) + fib(k-1)fib(n-k+2)$$

and equality holds if and only if $G \cong L_{n,k}$.

*Proof.* Let $u$ and $v$ be adjacent vertices on the cycle in $L_{n,k}$ such that $deg(u) = 2$ and $deg(v) = \Delta(L_{n,k})$. By Observation 1, Lemma 1 and Theorem 2,

$$i(L_{n,k}) = i(L_{n,k} - u) + i(L_{n,k} - N[u]) = fib(n+1) + fib(k-1)fib(n-k+2).$$

For $n \geq k \geq 3$ we consider an unicyclic graph $G$ of order $n$ that has $C_k$ as a subgraph such that $i(G)$ is minimal. Let $r := \max_{v \in V(G)}\{d(v, V(C_k))\}$.

Suppose that $G$ contains to end vertices $v_1$ and $v_2$ such that either the neighbor $n_1$ to $v_1$ is adjacent to the neighbor $n_2$ to $v_2$ or $n_1 = n_2$. Let $G'$ be the graph defined by $G' = (G - n_2v_2) + v_1v_2$. we observe that an independent set $S$ in $G'$ is independent in $G$ unless $\{v_2, n_2\} \subseteq S$. An independent set in $G$ containing both $v_1$ and $v_2$ is not independent i $G'$. Therefore

$$i(G) \geq i(G') - i(G' - N_{G'}[\{v_2, n_2\}]) + i(G - N_G[\{v_1, v_2\}]). \qquad (1)$$

Since $G' - N_{G'}[\{v_2, n_2\}]$ is an induced subgraph of $G - N_G[\{v_1, v_2\}]$ and $G' - N_{G'}[\{v_2, n_2\}] \ncong G - N_G[\{v_1, v_2\}]$ Theorem 1 gives that $i(G' - N_{G'}[\{v_2, n_2\}]) < i(G - N_G[\{v_1, v_2\}])$. Thus (1) implies $i(G) > i(G')$ which contradicts the choice of $G$.

The theorem is now proven by induction on $n-k$. Suppose first that $r \in \{0,1\}$ (This includes the case $n-k \in \{0,1\}$). If $r=0$ then $G \cong C_k$ and the result holds. Assume $r=1$. If $n=k+1$ then $G \cong L_{n,k}$ and the result holds for $G$. Therefore we assume that $n \geq k+2$.

For $n \leq 9$ the result can easily be verified in this case where $r=1$ and $n-k \geq 2$ (only seven graphs must be checked since we may assume nonexistence of two end vertices $v_1$, $v_2$ in $G$ such that the neighbor $n_1$ of $v_1$ is adjacent or equal to the neighbor $n_2$ of $v_2$).

Therefore, assume that $n \geq 10$. Since $r=1$ we may conclude that $k \geq 7$ by the observations done so far. Let $u \in V(G)$ such that $deg(u) \geq 3$. Both neighbors to $u$ in $C_k$ have valency two and for one of them, $v \in N(u) \cap V(C_k)$, we must have that $G - N[v] \ncong P_{n-4} \cup K_1$ and $G - v \ncong P_{n-1}$.

Since $G-v$ is a tree of order $n-1$ not isomorphic to $P_{n-1}$ and $G-N[v]$ is a forest of order $n-3$ with at least one isolated vertex it follows from Observation 1, Theorem 4 that $i(G-v) > fib(n+1)$ and $i(G-N[v]) \geq 2(2fib(n-4) + 3fib(n-7))$.

From Lemma 1 it follows that the theorem holds when $r=1$ if

$$4fib(n-4) + 6fib(n-7) \geq fib(n-k+2)fib(k-1)$$

when $n-k \geq 2$ and $k \geq 7$.

In the following we show by induction on $n-k \geq 1$ that

$$4fib(n-4) + 6fib(n-7) \geq fib(n-k+2)fib(k-1).$$

To prove this we only assume that $k \geq 7$ and $n > k$.

The case $n-k \in \{1,2\}$ is first considered. If $n-k=1$ we have

$$\begin{aligned} 4fib(n-4) + 6fib(n-7) &\geq 2fib(n-2) = fib(3)fib(n-2) \\ &= fib(n-k+2)fib(k-1). \end{aligned}$$

For $n-k=2$ we have

$$\begin{aligned} &4fib(n-4) + 6fib(n-7) = \\ &\qquad = 3fib(n-3) + fib(n-4) - 3fib(n-5) + 6fib(n-7) \\ &\qquad \geq 3fib(n-3) - 2fib(n-5) + fib(n-6) + 3fib(n-6) \\ &\qquad > 3fib(n-3) = fib(n-k+2)fib(k-1). \end{aligned}$$

Consider the case where $n-k=a \geq 3$ and assume that

$$4fib(n'-4) + 6fib(n'-7) \geq fib(n'-k+2)fib(k-1)$$

if $1 \le n' - k < a$. This assumption gives

$$\begin{aligned}4fib(n-4)+6fib(n-7) &= 4fib(n-5)+6fib(n-8)\\ &\quad +4fib(n-6)+6fib(n-9)\\ &\ge fib(n-k+1)fib(k-1)+fib(n-k)fib(k-1)\\ &= fib(n-k+2)fib(k-1).\end{aligned}$$

The case $r \in \{0,1\}$ is therefore done, so assume that $r \ge 2$ and that the result holds for each unicyclic graph $G'$ of order at most $n-1$ that contains $C_k$ as a subgraph. Let $v \in V(G)$ be a vertex in $G$ such that $d(v, V(C_k)) = r$. Since $G-v$ is an unicyclic graph of order $n-1$ and $G-N[v]$ is a spanning subgraph of an unicyclic graph with girth $k$ and $n-2$ vertices it follows from the assumptions, Lemma 1 and Theorem 1 that

$$i(G) = i(G-v) + i(G-N[v]) \ge i(L_{n-1,k}) + i(L_{n-2,k}) = i(L_{n,k}).$$

Further we must have that $G-v \cong L_{n-1,k}$ and $G-N[v] \cong L_{n-2,k}$ if $i(G) = i(L_{n,k})$. From these observations it can be concluded that $G \cong L_{n,k}$ if $i(G) = i(L_{n,k})$. □

The following theorem describes the unicyclic graph of order $n \ge 3$ with the fewest independent sets. This result was first proven by Vestergaard and Pedersen in [4]. Here is a simple proof for this theorem.

**Theorem 6.**
Let $G$ be an unicyclic graph of order $n$, then $i(G) \ge fib(n+1)+fib(n-1)$ and $i(G) = fib(n+1)+fib(n-1)$ if and only if $G \cong C_n$ or $G \cong L_{n,3}$.

*Proof.*
Let $u$ and $v$ be adjacent vertices on the cycle $C_k$ in $L_{n,k}$ such that $deg(v) = \Delta(L_{n,k})$. It follows that $G-u \cong P_{n-1}$, $G-N[u]$ is a spanning subgraph of $P_{n-3}$, and $G-N[u] \cong P_{n-3}$ if and only if $k=3$ or $k=n$. Let $G$ be a unicyclic graph and let $C_k$ denote the cycle in $G$. From the observations done so far, Theorem 1, Theorem 2, and Theorem 5 we have $i(G) \ge i(L_{n,k}) \ge i(P_{n-1}) + i(P_{n-3}) = fib(n+1)+fib(n-1)$ where equality holds if and only if $G \cong C_k$ or $G \cong L_{n,3}$. □

# 3 An upper bound for $i(G)$ depending on $|E(G)|$

This section gives an upper bound for $i(G)$ when $G$ is a connected graph of order $n$. First we introduce some graphs.

For a integer $n \geq 2$ and an integer $k \in \{0, \ldots, n-2\}$ we define $G_{n,k}$ as the graph of order $n$ obtained by addition of $k$ edges all incident to one common end vertex of the star $K_{1,n-1}$

In the article [4] the following bound for unicyclic graphs is proven. This bound can easily be proven by using Lemma 1, Theorem 1 and applying induction.

**Theorem 7.**
Let $G$ be an unicyclic graph of order $n$ and with girth at least $k \geq 3$. Then $i(G) \leq 2^{n-k} fib(k+1) + fib(k-1) \leq 1 + 3 \cdot 2^{n-2}$ and $i(G) = 1 + 3 \cdot 2^{n-2}$ if and only if $G \cong G_{n,1}$ or $G \cong C_4$.

To prove the desired bound the following lemma is used.

**Lemma 2.**
Let $H$ be a subgraph of a connected graph $G$ of order $n$. If $i(H) \leq 2^{|V(H)|-2} + 1$ then $i(G) \leq 2^{n-2} + 1$.

*Proof.* The theorem is proven by induction on $n - |V(H)|$. If $|V(H)| = n$ then by Theorem 1 $i(G) \leq i(H) \leq 2^{|V(H)|-2} + 1 = 2^{n-2} + 1$. If $|V(H)| < n$ then there exists an edge $xy \in E(G)$ such that $x \in V(H)$ and $y \notin V(H)$. Let $H'$ be the graph constructed from $H$ by adding the vertex $y$ and the edge $xy$. Since $i(H') = i(H' - y) + i(H' - N[y])$ we have by Theorem 1 that $i(H') \leq i(H) + (i(H) - 1) \leq 2 \cdot 2^{|V(H)|-2} + 1 = 2^{|V(H')|-2} + 1$. Since $H'$ is a subgraph of $G$ the induction hypothesis gives $i(G) \leq 2^{n-2} + 1$. □

An upper bound for $i(G)$ in terms of order and size can now be proven.

**Theorem 8.**
Let $G$ be a connected graph of order $n \geq 2$ and size $n - 1 + k$ then

$$i(G) \leq 1 + 2^{n-2-k} + 2^{n-2}$$

and equality holds if and only if $k \in \{0, \ldots, n-2\}$ and $G \cong G_{n,k}$ or $n \geq 3$ and $G \cong K_{2,n-2}$.

*Proof.*
It can easily be seen that

$$i(G_{n,k}) = 1 + 2^{n-2-k} + 2^{n-2}$$

and $i(K_{2,n-2}) = 3 + 2^{n-2} = 1 + 2^{n-2-k} + 2^{n-2}$ where $k = n - 3$.

The theorem shall be proven by induction on $k$. If $k \leq 1$ then $G$ is a tree or an unicyclic graph and the theorem is obtained from Theorem 3 or Theorem

7. Thus we may assume that $k \geq 2$ and that the results is true for $k' < k$. Since $k \geq 2$ $G$ must contain a cycle. Assume that $G$ contains $C_a$ as a subgraph and $a \geq 5$. By removing $k-1$ edges from $G$ we can get an unicyclic graph $G'$ such that $C_a$ is a subgraph of $G'$. By Theorem 7 we obtain that $i(G') \leq fib(4) + 2^{n-5} fib(6) = 3 + 2^{n-2}$. Since $G'$ is constructed by removing $k-1$ edges from $G$ we have $i(G) \leq i(G') - (k-1) \leq 3 + 2^{n-2} - (k-1) < 1 + 2^{n-2-k} + 2^{n-2}$.

Thus we may assume that each cycle in $G$ is isomorphic to $C_3$ or $C_4$. Assume that there exists a vertex $v \in V(G)$ such that $deg(v) = 2$ and $v$ is on a cycle in $G$. Since $G - N[v]$ is of order $n-3$ we have that $i(G - N[v]) \leq 2^{n-3}$ and from the induction hypothesis we obtain $i(G - v) \leq 1 + 2^{(n-1)-2-(k-1)} + 2^{(n-1)-2}$. From these observations we get the bound since

$$i(G) = i(G - v) + i(G - N[v]) \leq 1 + 2^{n-2-k} + 2^{n-2},$$

and if $i(G) = 1 + 2^{n-2-k} + 2^{n-2}$ we must have $G - N[v] \cong (n-3)K_1$ and either $G - v \cong K_{2,n-3}$ or $G - v \cong G_{n-1,k-1}$ where $k - 1 \in \{1, \ldots, (n-1) - 2\}$. Hence either $G \cong G_{n,k}$ for a $k \in \{2, \ldots, n-2\}$ or $G \cong K_{2,n-2}$. Thus we may assume that a vertex on a cycle in $G$ has at least three neighbors.

Assume that $G$ contains $C_4$ as an induced subgraph. Since each vertex on the cycle has at least three neighbors and each cycle of $G$ have length three or four $G$ must have one of the graphs from figure 1 as a subgraph. If $G'$ denotes one of these graphs then $i(G') \leq 2^{|V(G')|-2}$. From Lemma 2 we can conclude that the theorem holds for $G$ in this case.

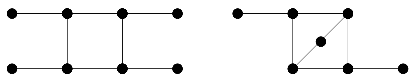

Figure 1: Illustration of a subgraph in $G$ when $G$ contains $C_4$ as an induced subgraph.

Thus we may assume that $C_4$ is not an induced subgraph of $G$ and since $k \geq 2$ we can conclude that one of the graphs $G_1, G_2$ or $G_3$ from figure 2 is a subgraph of $G$.

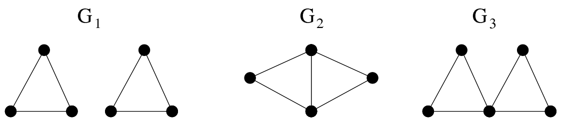


Figure 2: One of these graphs must be a subgraph of $G$.

Since $i(G_1) = i(K_3)^2 = 2^{|V(G_1)|-2}$ Lemma 2 implies that it can be assumed that $G_1$ is not a subgraph of $G$. If $G$ contains $G_2$ as a subgraph then $G$ must either have $K_4$ or the graph from figure 3 as a subgraph.

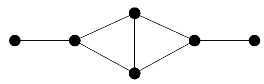

Figure 3: Illustration of a graph.

If $G'$ is one of these two graphs then $G'$ satisfies that $i(G') = 2^{|V(G')|-2} + 1$. Thus Lemma 2 gives that $i(G) \leq 2^{n-2} + 1$ in the case where $G_2$ is a subgraph of $G$.

Thus we may assume that $G$ contains $G_3$ as an induced subgraph. Since each vertex on a cycle in $G$ has at least three neighbors and no cycle in $G$ contains more than four vertices, $G$ must have the graph from figure 4 as a subgraph.

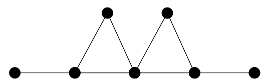

Figure 4: Illustration of a graph.

If $G'$ denotes this graph then $i(G') \leq 2^{|V(G')|-2}$. From this observation we obtain by Lemma 2 that $i(G) \leq 2^{n-2} + 1$ and the theorem has been proven in all cases. $\square$

# 4 Independent sets in graphs with minimum degree two or three

The following theorem gives an upper bound for $i(G)$ when $G$ is a connected graph with $\delta(G) \geq k$ for $k \in \{2, 3\}$. To obtain these bounds the following Lemma is used.

**Lemma 3.**
Let $H$ be a subgraph of a connected graph $G$. If there exists numbers $a$ and $b$ such that $i(G) \leq 2^{|V(H)|-a} + b$ and each vertex $v$ where $deg_G(v) \neq deg_H(v)$ is included in at least $b$ independent sets in $H$ then $i(G) \leq 2^{|V(H)|-a} + b$. If equality holds then $|E(G)| - |E(H)| = |V(G)| - |V(H)|$.

*Proof.* The theorem is proven by induction on $n - |V(H)|$. If $|V(H)| = n$ then by Theorem 1 $i(G) \leq i(H) \leq 2^{|V(H)|-a} + b = 2^{|V(G)|-a} + b$ and equality holds if and only if $G \cong H$. If $|V(H)| < n$ then there exists an edge $xy \in E(G)$ such that $x \in H$ and $y \notin H$. Let $H'$ be the graph constructed from $H$ by adding the vertex $y$ and the edge $xy$. Thus we obtain that $i(H') = i(H' - y) + i(H' - N[y]) = i(H) + i(H - x) = i(H) + (i(H) - i(H - N[x])) = 2i(H) - i(H - N[x]) \leq 2(2^{|V(H)|-a} + b) - b = 2^{|V(H')|-a} + b$. Thus the theorem follows by induction since each vertex $v$ satisfy $i(H' - N[v]) \geq i(H - N[v])$. $\square$

**Theorem 9.**
If $G$ is a graph of order $n$ such that $\delta(G) \geq 2$ then $i(G) \leq 2^{n-2} + 3$ and equality holds if and only if $G \cong C_5$ or $n \geq 4$ and $G \cong K_{2,n-2}$. If $G$ is not one of these graphs and $G$ cannot be obtained from $K_{2,n-2}$ by adding an edge between the two vertices of degree $n-2$ (let $K'_{2,n-2}$ denote this graph) then $i(G) < 2^{n-2} - 2^{n-4} + 7$.

*Proof.* The theorem is proven by induction on $|V(G)| + |E(G)|$. If $3 \leq n \leq 5$ it is easy to verify that the theorem is true. Let $G$ be a graph of order $n \geq 6$ and assume that the theorem is true for each graph $G'$ where $|V(G')| + |E(G')| < |V(G)| + |E(G)|$.

If $G$ has an edge $e$ incident with to vertices of degree at least three then the theorem follows by using the induction hypothesis on the graph $G - e$ if $G - e \notin \{C_5, K_{2,n-2}, K'_{2,n-2}\}$. If $G - e \in \{C_5, K_{2,n-2}, K'_{2,n-2}\}$ it is easily verified that $G$ satisfy the theorem.

Assume that $G$ contains a vertex $v$ such that $deg(v) = 2$ and its neighbors $v_1$ and $v_2$ satisfy that $deg(v_1) \geq 3$ and $deg(v_2) \geq 3$. In this case the induction hypothesis gives $i(G - v) \leq 2^{n-3} + 3$. Since $i(G - N[v]) \leq 2^{n-3}$ we obtain that $i(G) = i(G-v) + i(G-N[v]) \leq 2^{n-2} + 3$. If $i(G) = 2^{n-2} + 3$ then $G - N[v]$ must be isomorphic to $(n-3)K_1$ and it can be concluded that $G \cong K_{2,n-2}$. If $G - v$ is not one of the graphs described in the theorem then $i(G - v) < 2^{n-3} - 2^{n-5} + 7$ and $i(G - N[v]) \leq 2^{n-3} - 2^{n-5}$ since $G - N[v] \ncong (n-3)K_1$. Thus it follows that $i(G) = i(G - v) + i(G - N[v]) < 2^{n-3} - 2^{n-5} + 7$. Since it can easily be checked that $i(G) < 2^{n-2} - 2^{n-4} + 7$ if $G - v$ is one of the graphs from the theorem and $G$ is not one of these graphs the theorem follows in this case where there exists a vertex only with neighbors of degree at least three.

Thus we may assume that $G$ has the following properties:

- If $uw \in E(G)$ then $deg(u) = 2$ or $deg(w) = 2$.
- If $u \in V(G)$ and $deg(u) = 2$ then $u$ has a neighbor $w$ such that $deg(w) = 2$.

If $G$ is not connected the theorem can be obtained by using the induction hypothesis and the fact that a component with three vertices has 4 independent sets.

Since it easily can be verified that each cycle $C_n$ where $n \geq 6$ satisfies the theorem we may assume that there exists at vertex $v$ with neighbors $v_1$ and $v_2$ such that $deg(v) = deg(v_1) = 2$ and $deg(v_2) \geq 3$.

In the following we show that it can be assumed that some graphs can not be a subgraph of $G$ when a set of vertices from $G$ must be specific vertices in the subgraph. A combination of such a graph and a vertex set will be called a configuration.

First we assume that $v_1$ is adjacent to $v_2$. If $deg(v_2) = 3$ then the graph from figure 5 is a subgraph of $G$. Since this graph $G'$ satisfies that $i(G') = 2^{|V(G')|-2} - 2^{|V(G')|-4} + 5$ it can be concluded from Lemma 3 that $i(G) \leq 2^{n-2} - 2^{n-4} + 5 < 2^{n-2} - 2^{n-4} + 7$ since each vertex in $G'$ is contained in at least five independent sets.

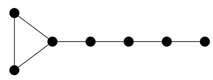

Figure 5: Illustration of subgraph of $G$ when $deg(v_2) = 3$ and $v_1v_2 \in E(G)$.

Thus we may assume that $deg(v_2) \geq 4$. It follows that one of the configurations $G_1$, $G_2$ or $G_3$ from figure 6 is in $G$.

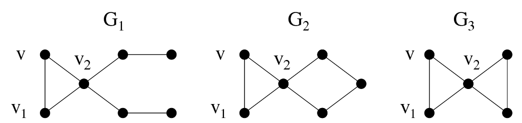


Figure 6: Illustration of the configurations $G_1, G_2$ and $G_3$.

Assume that $G_1$ is in $G$. If $G_1$ is a spanning subgraph of $G$ then $i(G) < i(G_1) = 2^{n-2} - 2^{n-4} + 7$ and otherwise one of the graphs form figure 7 must be a subgraph of $G$.

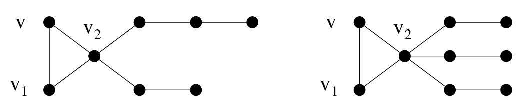


Figure 7: Illustration of the configurations in $G$ when $G_1$ is in $G$ as a non-spanning subgraph.

By applying Lemma 3 to these graphs it follows that $i(G) < 2^{n-2} - 2^{n-4} + 7$. Thus it can be assumed that $G_1$ is not in $G$. If $G_2$ is a subgraph of $G$ then either the graph $G'$ illustrated in figure 5 or the graph $G''$ obtained by adding vertices adjacent to both $a$ and $v_2$ in $G_2$ is a spanning subgraph of $G$. As earlier in the proof the theorem follows when $G'$ is in $G$ and if $G''$ is a spanning subgraph of $G$ then $i(G'') \leq i(G') = 5 + 3 \cdot 2^{n-4} < 2^{n-2} - 2^{n-4} + 7$. Thus it is assumed that $G_3$ is in $G$ and $G_1$ and $G_2$ is not in $G$. In this case we obtain a contradiction since $n \geq 6$ and it must hold $G \cong G_3$.

Thus it can be assumed that $v_1$ and $v_2$ are not adjacent. From this assumption we have that $G$ contains one of the configurations $G_4$ or $G_5$ from figure 8.

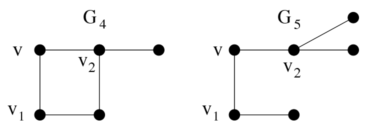


Figure 8: Illustration of configuration $G_4$ and $G_5$.

Assume that $G_4$ is in $G$. Since each neighbor to $v_2$ in $G$ has exactly two neighbors then one of the configurations $G_6$ or $G_7$ from figure 9 is in $G$.

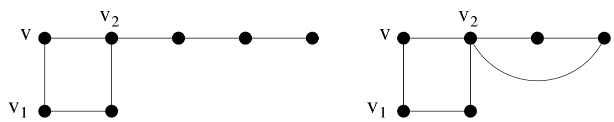


Figure 9: Illustration of configurations.

If one of these is a spanning subtree the theorem easily follows, and otherwise one of the graphs from figure 10 must be a subgraph of $G$.

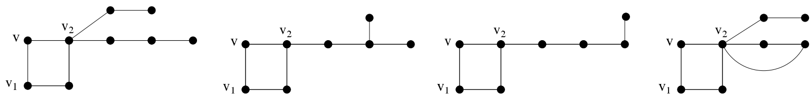

Figure 10: Illustration of configurations.

By applying Lemma 3 to these subgraphs of $G$ we obtain that $i(G) < 2^{n-2} - 2^{n-4} + 7$. From this observation we may assume that $G_5$ is in $G$. Let $x$ and $y$ denote two vertices in $N(v_2)\backslash\{v\}$. We show that the vertices $x$ and $y$ only have $v_2$ as a neighbor in the configuration $G_5$. If $x$ and $y$ are adjacent then $x$ and $y$ could have been chosen as $v$ and $v_1$ and thus the theorem follows in this case.

If $x$ or $y$ are adjacent to the vertex from $N(v_1)\backslash\{v\}$ the graph $G'$ from figure 11 must be a subgraph of $G$. Since $i(G') \leq 2^{|V(G')|-2} - 2^{|V(G')|-4} + 6$ and each vertex in $G'$ is included in at least six independent sets it follows from Lemma 3 that $i(G) < 2^{n-2} - 2^{n-4} + 7$.

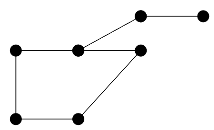

Figure 11: Illustration of $G_8$.

Thus we may assume neither $x$ nor $y$ have more than one neighbor in the $G_5$-subgraph. Thus $G$ must either have one of the graphs from figure 12(a) as a subgraph and $deg(v_2) = 3$ or one of the graphs from 12(b) is a subgraph of $G$.

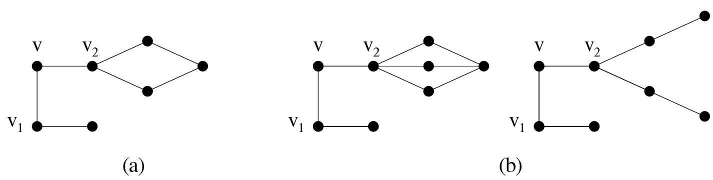


Figure 12: Illustration of graphs.

By using Lemma 3 it follows that $i(G) < 2^{n-2} - 2^{n-4} + 7$ in all three cases and the theorem is proved. $\square$

By using Theorem 9 a upper bound for $i(G)$ can easily be found when $G$ is a graph with minimum degree three.

**Theorem 10.**
Let $G$ be a graph of order $n$ such that $\delta(G) \geq 3$ then $i(G) \leq 2^{n-3} + 7$, and equality hold if and only if $G \cong K_{3,n-3}$.

*Proof.* The theorem is proven by induction on $n+e$. If $n = 4$ it follows that $G \cong K_4$ and $i(G) = 5 \leq 2^{n-3} + 7$. thus it can be assumed that $G$ is a graph of order $n \geq 5$ and each graph $G'$ where $\delta(G') \geq 3$ and $|V(G')| + |E(G')| < n + e$ satisfy that $i(G') \leq 2^{|V(G')|-3} + 7$ and equality hold if and only if $G' \cong K_{3,|V(G')|-3}$.

Let $v$ be a vertex in $G$ such that $deg(v) = \Delta(G) \geq 3$. If $\delta(G - v) \geq 3$ then it follows that $i(G-v) \leq 2^{n-4}+7$ and $i(G-N[v]) \leq 2^{n-4}$. From these inequalities it follows that $i(G) \leq 2^{n-3} + 7$ and if equality holds then $G - v \cong K_{3,n-4}$ and $G - N[v] \cong (n-4)K_1$ which implies that $G \cong K_{3,n-3}$.

Thus it can be assumed that $\delta(G - v) = 2$. If $G - v$ is one of the graphs described in Theorem 9 it can easily be checked that $i(G) \leq 2^{n-3} + 7$ and equality only holds when $G \cong K_{3,n-3}$. Thus it follows by Theorem 9 that it can be assumed that $i(G-v) < 2^{n-3} - 2^{n-5} + 7$. Since $i(G) = i(G-v) + i(G-N[v])$ the theorem follows if $i(G - N[v]) \leq 2^{n-5}$. If $G' \in \{P_4, 3K_2, K_2 \cup P_3\}$ is a subgraph of $G - N[v]$ then $i(G - N[v]) \leq 2^{n-4-|V(G')|} i(G') \leq 2^{n-5}$. Thus it can be assumed that $G - N[v]$ consists of isolated vertices and a graph isomorphic to $2K_2$ or a star $K_{1,a}$. Since $\Delta(G) = \delta(G) = 3$ the graph $2K_2$ can not be a subgraph of $G - N[v]$. Similar it follows that if $K_{1,a}$ is a subgraph of $G - N[v]$ then $a \leq 3$. By verification the theorem easily follows in this case. $\square$

## References


[1] M. J. Jou and G. J. Chang. Survey on counting maximal independent sets, in: Proceedings of the Second Asian Mathematical Conference. *World Scientific, 265-275*, 1995.

[2] M. J. Jou and G. J. Chang. The number of maximum independent sets in graphs. *Taiwanese Journal of Mathematics, Vol 4 Nr. 4, 685-695*, 2000.

[3] S. B. Lin and C. Lin. Trees and forests with large and small independent indices. *Chinese Journal of Mathematics, Vol 23 Nr. 3, 199-210*, 1995.

[4] A. S. Pedersen and P. D. Vestergaard. On the number of independent sets in unicyclic graphs. *preprint.*

[5] H. Prodinger and R. F. Tichy. Fibonacci numbers of graphs. *Fibonacci Quart., Vol. 20, 16-21*, 1982.

# Total Well Dominated Trees

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**Abstract**

Let $G = (V, E)$ be a graph with no isolated vertex. A set $D$ is called total dominating in $G$ if each vertex in $G$ is adjacent to a vertex from $D$, and $D$ is a minimal total dominating set if any subset $D' \subset D$ is not a total dominating set in $G$. If all minimal total dominating sets in $G$ have the same cardinality then $G$ is a total well dominated graph. In this paper we study composition and decomposition of total well dominated trees.

**Keywords:** total domination, decomposition/composition, total well dominated
**AMS subject classification: 05C69**

# 1 Notation

For notation and graph theory terminology we in general follow [1]. Specifically, let $G = (V, E)$ be a graph with vertex set $V$ and edge set $E$. A *dominating set* of $G$ is a set $D$ of vertices of $G$ such that every vertex in $V \backslash D$ is adjacent to a vertex in $D$. Further, if also each vertex in a dominating set $D$ is adjacent to a vertex from $D$ then $D$ is a *total dominating set*. The *total domination number* of $G$, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set. If a total dominating set $D$ satisfies that no proper subset of $D$ is a total dominating set then $D$ is a *minimal total dominating set*. The *upper total domination number* of a graph $G$, denoted by $\Gamma_t(G)$, is the maximum cardinality of a minimal total dominating set. If all minimal total dominating sets in a graph $G$ have the same cardinality the graph is called *total well dominated* (or just $TWD$). Thus a graph $G$ is TWD if and only if $\gamma_t(G) = \Gamma_t(G)$. For a set $D$ and a vertex $x \in D$ the *private neighborhood* of $x$ is defined by $pn(x, D) := \{y \in N[x] | N[y] \cap D = x\}$. A vertex of degree one is called a *leaf* and a vertex adjacent to a leaf is called a *stem*.

For two vertices $x$ and $y$ in a graph $G$ we denote the distance between the vertices by $d_G(x, y)$. Further if $S \subseteq V(G)$ we define $d_G(x, S) := min_{s \in S}\{d_G(x, s)\}$.

If $G$ is a graph and $S$ is a vertex set in $G$, then the induced subgraph of $G$ with vertex set $S$ is denoted $G[S]$.

For $k \geq 1$ let $A_k$ be the graph with vertex set $V(A_k) = \{x_1, x_2, \ldots, x_{k+1}, x, y\}$ and edge set $E(A_k) = \{x_1x_2, x_2x_3, \ldots, x_kx_{k+1}, x_kx, x_{k+1}y\}$. Thus $A_k$ is the graph illustrated in figure 1.

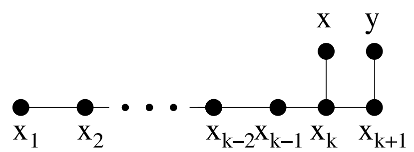


Figure 1: Illustration of $A_k$.

Further let $\mathcal{A}_4$ be the family of graphs with the structure illustrated in figure 2 (a) and let $\mathcal{A}_5$ be the family of graphs with the structure illustrated in figure 2(b).

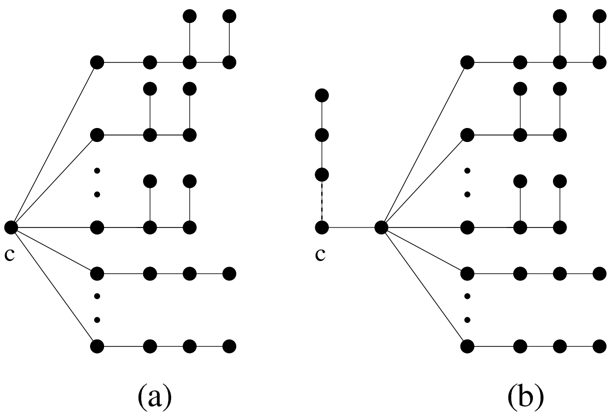


Figure 2: (a) illustrates $\mathcal{A}_4$ and (b) illustrates $\mathcal{A}_5$. In (b) the doted line/edge indicates that there may or may not be an edge and a $P_3$-subgraph.

In $A_i$ we call the vertex $x_1$ an attachment vertex and in a graph from $\mathcal{A}_4$ or $\mathcal{A}_5$ we call the vertex $c$ an attachment vertex. Further in a path $P_n$ a vertex with smallest degree is called an attachment vertex.

For one of these graphs $H$ with attachment vertex $a$ and a graph $G$ with a vertex $v$ we define the graph obtained by attaching $H$ to $v$ in $G$ as the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{av\}$. In the obtained graph we say that $H$ is attached at $v$.

## 2 Decomposition/composition-rules

In this section we give the main decomposition/composition-rules that are used for total well dominated tress. Each rule is of the kind saying that if $G$ is a graph with some special structure then $G$ is TWD if and only if some subgraphs of $G$ is TWD. If a graph does not have the structure as the graph $G$ we say that the graph is reduced by the rule (or the lemma with the rule).

For the decomposing/composing we use some special vertices and admissible sets, therefor these concepts is defined in the following.

**Definition 1** *Let $G$ be a TWD graph and let $v \in V(G)$. If the graph $G' = (V(G) \cup \{x\}, E(G) \cup \{vx\})$ is TWD and $\gamma_t(G) = \gamma_t(G')$ then $v$ is called a* special vertex.

**Definition 2** *Let $G$ be a graph and let $S \subseteq V(G)$. The set $S$ is called* admissible *if $G[S]$ and $G - N[S]$ have no isolated vertices and each vertex $v \in S$ is a stem in $G[S]$ or $pn(v, S \cup (V(G) \backslash N[S])) \neq \emptyset$.*

From the definition of an admissible set it can be seen that a set $S$ is admissible in $G$ if $S \cup (V(G) \backslash N[S])$ is a total dominating set in $G$ and $(S \cup (V(G) \backslash N[S])) \backslash \{s\}$ is not a total dominating set in $G$ for any vertex $s \in S$.

**Observation 1** *If $S$ is an admissible set in a graph $G$, then for each minimal total dominating set $S'$ in $G - N[S]$, the set $S \cup S'$ is a minimal total dominating set in $G$. Thus if $G$ is TWD then $G - N[S]$ is TWD when $S$ is admissible.*

**Lemma 1** *A graph $G$ is TWD if and only if each component of $G$ is TWD.*

**Lemma 2** *If a vertex $v \in V(G)$ is adjacent to two leaves $l_1$ and $l_2$ then $G$ is TWD if and only if $G - l_1$ is TWD.*

**Lemma 3** *Let $G$ be a graph containing two adjacent stems $s_1$ and $s_2$. If $L$ denotes the leaves adjacent to $s_1$ or $s_2$ and $C_1, \ldots, C_k$ are the components of $G - (\{s_1, s_2\} \cup L)$ then $G$ is TWD if and only if $G_i := G - \{C_1, C_2, \ldots, C_{i-1}, C_{i+1}, C_{i+2} \ldots, C_k\}$ is TWD for each $i \in \{1, \ldots, k\}$.*

**Proof.** The statement is trivial for $k \leq 1$, so assume $k \geq 2$. Let $D_i$ denote a minimal total dominating set in $G_i$ then $\bigcup_{i=1}^{k} D_i$ is a total dominating set in $G$, further it must be minimal since each total dominating set of $G$ must contain the stems $s_1$ and $s_2$ and $D_i$ is a minimal total dominating set in $G_i$ for $i \in \{1, \ldots, k\}$. Since $|\bigcup_{i=1}^{k} D_i| = \sum_{i=1}^{k} |D_i| - (k-1) * 2$ it follows that $G_i$ is TWD if $G$ is TWD. Now let $D$ be a minimal total dominating in $G$ and let $D_i := D \cap V(G_i)$. Since $D$ contains the stems $s_1$ and $s_2$ and $D$ is minimal total dominating in $G$ the set $D_i$ must be a minimal total dominating set of $G_i$. Since $|D| = \sum_{i=1}^{k} |D_i| - (k-1) * 2$ the graph $G$ is TWD if $G_i$ is TWD for each $i \in \{1, \ldots, k\}$. $\square$

**Lemma 4** *Let $G$ be a graph with a path $l_1, s_1, v_1, v_2, s_2, l_2$ and assume that $deg(l_1) = deg(l_2) = 1$. Then $G$ is TWD if and only if $G - v_1v_2$ is TWD.*

**Proof.** Since $s_1$ and $s_2$ are stems these vertices must be in all total dominating sets in $G$ and $G - v_1v_2$. Thus a total dominating set of $G$ is also a total dominating set in $G - v_1v_2$ and a total dominating set of $G - v_1v_2$ is trivially a total dominating set of $G$. Thus the result follows. $\square$

**Lemma 5** *Let $T$ be a TWD tree reduced by Lemma 2 and let $s$ be a stem of $T$ having valency at least 3. Then $T$ contains another stem at distance at most 3 from $s$.*

**Proof.** Let $s$ be a stem in $T$ and assume that $deg(s) \geq 3$ and that no other stem in $T$ is within distance 3 from $s$. Since $s$ is adjacent to exactly one leaf $l$ there must exists a path $P : a, b, c, s, d, e, f$ in $T$. For each vertex $x$ from $P$ let $C_x$ denote the union of all components of $T - x$ not containing vertices from $P$.

Since no stem $s' \neq s$ is within distance 3 from $s$ the set $V(C_x)\backslash N[x]$ is a total dominating set for $C_x$ when $x \in \{b, e\}$. Thus there is a minimal total dominating set $D_x$ of $C_x$ not containing a vertex adjacent to $x$.

For $x \in \{c, d\}$ let $D_x^+$ be a minimal total dominating set of $C_x$ and $D_x^-$ be a minimal total dominating set of $C_x - N[x]$.

For $x \in \{a, f\}$ let $y$ be the vertex from $P$ adjacent to $x$. Further let $D_x^+$ be a minimal total dominating set of $G[V(C_x) \cup \{x, y\}]$ not containing $y$ and $D_x^-$ be a minimal total dominating set of $G[V(C_x) \cup \{x\}]$ not containing $x$.

Further let $D_s$ be a minimal total dominating set of $C_s - N[s]$ and let $S := D_s \cup D_b \cup D_e$.

Now let $A := \{c, s, d\} \cup D_a^- \cup D_c^- \cup D_d^- \cup D_f^- \cup S$, $B := \{s, d\} \cup D_a^+ \cup D_c^+ \cup D_d^- \cup D_f^- \cup S$, $C := \{c, s\} \cup D_a^- \cup D_c^- \cup D_d^+ \cup D_f^+ \cup S$ and $D := \{s, l\} \cup D_a^+ \cup D_c^+ \cup D_d^+ \cup D_f^+ \cup S$. By construction all of these sets are minimal total dominating sets, and since $T$ is TWD $|A| = |B| = |C| = |D|$. Since $|A| = |B|$ we obtain $|D_a^-| + |D_c^-| + 1 = |D_a^+| + |D_c^+|$ and since $|A| = |C|$ we obtain $|D_d^-| + |D_f^-| + 1 = |D_d^+| + |D_f^+|$. But then

$$\begin{aligned} |D| &= 2 + |D_a^+| + |D_c^+| + |D_d^+| + |D_f^+| + |S| \\ &= 4 + |D_a^-| + |D_c^-| + |D_d^-| + |D_f^-| + |S| = |A| + 1. \end{aligned}$$

Thus we obtain a contradiction. □

**Lemma 6** *Let $G$ be a graph with a vertex $v$ adjacent to two stems $s_1$ and $s_2$. If $G$ is reduced by Lemma 2, 3 and 4 then $G \cong P_3 \circ K_1$ or $G$ is not TWD.*

**Proof.** Assume $G$ is TWD and reduced by Lemma 2, 3 and 4. Let $l_i$ be a leaf adjacent to $s_i$ for $i \in \{1, 2\}$. First assume that $A := \{s_1, s_2, l_1, l_2\}$ is an admissible set. Then there is a minimal total dominating set $D$ such that $A \subseteq D$. But then $(D\backslash\{l_1, l_2\}) \cup \{v\}$ is a total dominating set of smaller cardinality. Thus it can be assumed that $A$ is not admissible and thus $s_1$ or $s_2$ must be adjacent to a stem. Since $G$ is reduced by Lemma 4 that stem must be $v$, and since $G$ is reduced by Lemma 2 and Lemma 3 we obtain that $G \cong P_3 \circ K_1$. $\square$

**Lemma 7** *Let $G$ be a graph and let $v \in V(G)$. If $G_i$ is the graph obtained from $G$ by attaching $i$ $P_3$'s to the vertex $v$ then $G_1$ is TWD if and only if $G_2$ is TWD.*

**Proof.** Let $v_1, v_2, v_3$ and $u_1, u_2, u_3$ be the two $P_3$'s added to $G$ to obtain $G_2$ and assume that $\{v_1v, u_1v\} \subseteq E(G_2)$. Since $\{u_2, u_3\}$ is an admissible set in $G_2$ it follows that $G_1 = G_2 - N[\{u_2, u_3\}]$ is TWD if $G_2$ is TWD. Now assume $G_1$ is TWD and let $D$ be any minimal total dominating set in $G_2$. Assume WLOG that $d_{G_2}(v, \{v_1, v_2, v_3\} \cap D) \leq d_{G_2}(v, \{u_1, u_2, u_3\} \cap D)$ then $D \backslash \{u_1, u_2, u_3\}$ is a minimal total dominating set in $G_2 - N_{G_2}[\{u_2, u_3\}] \cong G_1$. Since $|D \cap \{u_1, u_2, u_3\}| = 2$ and $G_1$ is TWD it follows that $|D| = \gamma_t(G_1) + 2$ and thus $G_2$ is TWD. $\square$

**Lemma 8** *Let $G$ be a connected graph with a $P_4$ attached at a vertex $v$ and let $H$ be the graph obtained by removing the attached $P_4$. If $v$ is adjacent to a stem in $G$ then $G$ is TWD if and only if $H$ is TWD and $v$ is special in $H$.*

**Proof.** First assume that $G$ is TWD and let $v_1, v_2, v_3, v_4$ be the $P_4$ attached at $v$ such that $vv_1 \in E(G)$. Since $\{v_2, v_3\}$ and $\{v_3, v_4\}$ are admissible sets $G - N[\{v_2, v_3\}]$ and $H = G - N[\{v_3, v_4\}]$ is TWD and further $v$ is special. Now assume that $H$ is TWD and $v$ is a special vertex in $H$ and consider a minimal total dominating set $D$ in $G$. If $\{v_3, v_4\} \subseteq D$ then $D \backslash \{v_3, v_4\}$ is just a minimal total dominating set in $G - \{v_2, v_3, v_4\}$ and since $v$ is special we obtain $|D| = 2 + \gamma_t(H)$ in this case. Otherwise $D \cap \{v_1, v_2, v_3, v_4\} = \{v_2, v_3\}$ since $v$ is adjacent to a stem in $G$. We see that $D \backslash \{v_2, v_3\}$ is a minimal total dominating set in $H$ and $|D| = 2 + \gamma_t(H)$. Since $H$ is TWD it follows that $G$ is TWD. $\square$

**Lemma 9** *Let $G$ be a graph with a $P_2 : v_1, v_2$ attached at a non stem $v \in V(G)$ such that $vv_1 \in E(G)$ and $deg(v) \geq 3$. Let $C_1, \ldots, C_k$ be the components of $G - \{v, v_1, v_2\}$. If $G_i$ denotes the graph $G[V(C_i) \cup \{v, v_1, v_2\}]$ for $i \in \{1, \ldots, k\}$ then $G$ is TWD if and only if $G_1, \ldots, G_k$ are TWD.*

**Proof.** Let $D$ be a minimal total dominating set in $G$. If $v \in D$ then $v$ is the only vertex that dominates $v_1$ and it follows that $D \cap G_i$ is a minimal total dominating set in $G_i$. Since $|D \cap \{v, v_1, v_2\}| = 2$ it follows that $G$ is TWD if $G_1, \ldots, G_k$ are TWD. Assume that $G$ is TWD and let $D_i$ be a minimal total dominating set in $G_i$. Since $v$ is not a stem $G$ has a minimal total dominating set $D$ such that $D \cap G_i = D_i$. If $v_2 \in D_i$ then $(D \cup \{v\}) \backslash \{v_2\}$ must be a minimal total dominating set in $G$ since $G$ is TWD and therefor $(D_i \cup \{v\}) \backslash \{v_2\}$ must be a minimal total dominating set in $G_i$. Thus it follows that $G_i$ is TWD if all of the minimal total dominating sets in $G_i$ not containing $v_2$ have the same cardinality. Assume that $D_i$ and $D_i'$ are minimal total dominating sets in $G_i$ not containing $v_2$. Since $G$ is TWD and the sets $D$ and $(D \backslash D_i) \cup D_i'$ are minimal total dominating sets in $G$ it follows that $|D_i| = |D_i'|$. Thus $G_1, \ldots, G_k$ are TWD if $G$ is TWD. □

**Lemma 10** *Let $T$ be a tree reduced by Lemma 4 and 8 with a vertex $v$ such that all components of $T - v$ except one namely $C$, $C \ncong P_1$, are components isomorphic to $P_4$ or $A_2$ attached at $v$ and at least one $P_4$ is attached at $v$. Let $x$ denote the vertex from $V(C) \cap N[v]$, let $C_1, \ldots, C_k$ be the components of $C - N[x]$ and let $v_i$ denote the vertex in $C_i$ adjacent to a vertex from $C - C_i$. Then $G$ is TWD if and only if each of $C_1, \ldots, C_k$ are TWD, $v_i$ is a special vertex in $C_i$ adjacent to a stem in $C_i$ and $C_i \ncong P_2$.*

**Proof.** First assume that $T$ is TWD. Let $x_1, x_2, x_3, x_4, x_5 = v, x_6 = x, x_7 \ldots, x_a$ be a path in $T$ such that $x_1, x_2, x_3, x_4$ is a $P_4$ attached at $v$ and $x_4 v \in E(G)$. Since $G$ is reduced by Lemma 8 the vertex $x$ cannot be a stem. Neither is $x_7$ a stem, for assume otherwise that $x_7$ is a stem and let $D$ be a minimal total dominating set in the component of $T - xx_7$ containing $x_7$. Since $x_7$ is a stem $D$ is an admissible set in $T$ containing $x_7$ and by Observation 1 all components of $T - N[D]$ is TWD. But the component of $T - N[D]$ containing $v$ is not TWD so we have a contradiction. Thus we may assume that $x_7$ is not a stem and therefore $\{v, x\}$ is an admissible set. Since $T - N[\{v, x\}]$ contains $C_1, \ldots, C_k$ as components each of these is TWD. Let $C_i$ be the component containing $x_8$, we shall show that $C_i$ has a stem adjacent to $x_8$ and that $C_i \ncong P_2$. Assume otherwise that either $C_i \cong P_2$ or $C_i$ does not have a stem adjacent to $x_8$. Let $C'$ be the component of $T - xx_7$ containing $x_7$. By the assumptions the set $D'' := V(C') \backslash (N(x_8) \cap V(C_i))$ is a total dominating set for $C'$. Let $D'$ be a minimal total dominating set of $C'$ such that $D' \subseteq D''$. Since $N(x_8) \cap D'' = \{x_7\}$ the vertex $x_7$ must be in $D'$ and $D'$ is an admissible set in $T$. But the component of $T - N[D']$ containing $v$ is not TWD. This contradiction proves that in $C_i$ there is a stem adjacent to $x_8$ and that $C_i \ncong P_2$. Thus it just remains to prove that $x_8$ is a special vertex in $C_i$. Since $T$ is reduced by Lemma 4 and $x_8$ is adjacent to a stem in $C_i$ the vertex $x_7$ cannot be adjacent to a stem in $G - C_i$. Since $\{x_1, x_2, v, x\}$ and $\{x_1, x_2, x_4, v\}$ are admissible sets all components of $C - x$ must be TWD and for each such component $H$ the graph obtained by removing the vertex adjacent to $x$ must be TWD and have the same total domination number as $H$.

Let $D$ be a minimal total dominating set in $C' \cap (C_1 \cup C_2 \cup \cdots \cup C_{i-1} \cup C_{i+1} \cup C_{i+2} \cup \cdots \cup C_k)$ not containing any of the vertices $v_1, \ldots, v_k$. Now $D \cup \{v, x\}$

and $D\cup\{x_4, v\}$ are admissible sets and thus all components of $T-N[D\cup\{v, x\}]$ and $T - N[D \cup \{x_4, v\}]$ is TWD. Since $|D \cup \{v, x\}| = |D \cup \{x_4, v\}|$ we have $\gamma_t(T-N[D\cup\{x_4, v\}]) = \gamma_t(T-N[D\cup\{v, x\}])$ and by considering the components of these graphs we obtain that $C_i$ and the graph obtained by attaching a $P_1$ to $x_8$ in $C_i$ must be TWD and have the same total domination number, i.e. $x_8$ is special in $C_i$.

Now assume that $T$ can be constructed as described in the lemma and let $D$ be a minimal total dominating set in $T$. Since $v_1, \ldots, v_k$ are adjacent to a stem in $C_1, \ldots, C_k$ the set $D \cap C_i$ is a minimal total dominating set in $C_i$ or the graph obtained from $C_i$ by attaching a $P_1$ to $v_i$. Now consider the set $D' := D\backslash\{V(C_1) \cup \cdots \cup V(C_k)\}$. If this does not contain isolated vertices it is a minimal total dominating set in $T - V(C_1) - \cdots - V(C_k)$. Otherwise $D'$ contains exactly one vertex $y$ from $N(x)\backslash\{v\}$, $\{x, v\} \cap D' = \emptyset$ and $D'\backslash\{y\}$ is a minimal total dominating set in the component of $T - x$ containing $v$. By considering minimal total dominating sets not containing $v$ in this component it can be observed that they all have cardinality $\gamma_t(T - C_1 - \cdots - C_k) - 1$ and thus we obtain that $G$ is TWD. $\square$

**Corollary 1** *Let $T$ be a TWD tree reduced by Lemma 2, 3, 4, 8, 9 and 10. If $T$ has a leaf $v$ then $T \in \{P_2, P_4, P_3 \circ K_1\}$ or $T$ has the structure as one of the graphs form figure 3 where $v$ is a leaf in $T - T'$.*

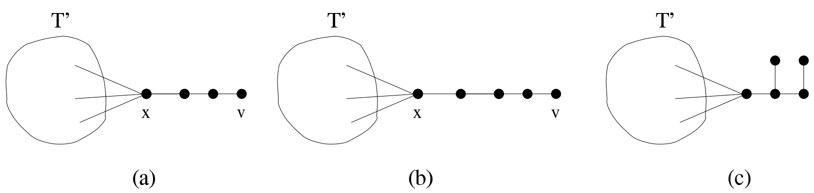


Figure 3: Illustration of structure near leaf. $deg(x) \geq 3$ in (a) and (b) .

**Lemma 11** *Let $G$ be a graph with two $A_2$'s attached at a vertex $v$ and let $H$ be the graph obtained by removing one of the attached $A_2$ graphs. Then $G$ is TWD if and only if $H$ is TWD.*

**Proof.** Let $v, v_1, v_2, v_3, v_4$ and $v, u_1, u_2, u_3, u_4$ be paths in $G$ such that $v_1$ and $u_1$ are contained in different $A_2$ graphs attached at $v$. Since $\{v_2, v_3\}$ is an admissible set in $G$ we obtain that the graph $H = G - N[\{v_2, v_3\}]$ is TWD if G is TWD. Conversely, let $D$ be a minimal total dominating set $D$ in $G$. Since $v_1$ and $u_1$ cannot both be in $D$ we may assume that $v_1 \notin D$. Thus the $A_2$ attached at $v$

containing $v_1$ contains exactly two vertices $v_2, v_3$ from $D$. The set $D \backslash \{v_2, v_3\}$ is then a minimal total dominating set of $H = G - N[\{v_2, v_3\}]$. Thus if $H$ is TWD then $G$ must also be TWD. □

**Lemma 12** *Let $G$ be a graph with two $A_3$'s attached at a non-stem $v$ with $deg(v) \geq 3$. Then $G$ is not TWD.*

**Proof.** Let $v, v_1, v_2, v_3, v_4, v_5$ and $v, u_1, u_2, u_3, u_4, u_5$ be paths in $G$ such that $v_1$ and $u_1$ are contained in different $A_2$ graphs attached at $v$. Let $D := V(G) \backslash \{v, v_1, u_1\}$ then $D$ is a total dominating set of $G$ since $v$ is not a stem and $deg(v) \geq 3$. Let $D'$ be a minimal total dominating set such that $D' \subset D$. Since $D'' := (D \backslash \{v_2, u_2\}) \cup \{v\}$ is a total dominating set and $|D''| < |D'|$ the graph cannot be TWD. □

**Lemma 13** *Let $H$ be a graph with a path $P : v_1, v_2, v_3$ such that $deg(v_2) = deg(v_3) + 1 = 2$. If $G$ is the graph obtained from $H$ by attaching a $A_1$ to $v_1$ and $v_3$ then $G$ is TWD if and only if $H$ is TWD.*

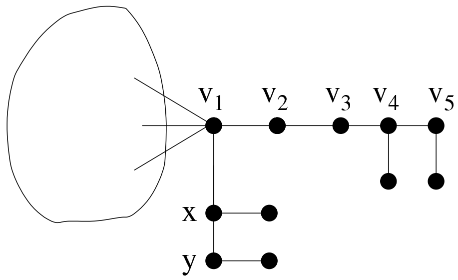


Figure 4: Illustratition of $G$.

**Proof.** In the following we use the notation as illustrated in figure 4. Let $D$ be a minimal total dominating set in $G$. It follows that $v_2 \notin D$ and $(D \cap V(H)) \cup \{v_2\}$ is a minimal total dominating set for $H$ and $|(D \cap V(H)) \cup \{v_2\}| = |D| - 3$. Thus $G$ is TWD if $H$ is TWD.

Conversely let $D$ be a minimal total dominating set in $H$. Since $v_2$ is a stem in $H$ we have $v_2 \in D$. Consider the set $D' := (D \backslash \{v_2\}) \cup \{x, y, v_4, v_5\}$. This set is a minimal total dominating set and $|D'| = |D| + 3$. Thus $H$ is TWD if $G$ is TWD. □

**Lemma 14** *Let $H$ be a graph and let $v \in V(H)$. Now let $G$ be a graph obtained from $H$ by attaching $A_1$ and a graph from $\mathcal{A}_4$ to $v$ and let $G'$ be the graph obtained from $H$ by attaching $P_2$ to $v$. Then $G$ is TWD if and only if $G'$ is TWD.*

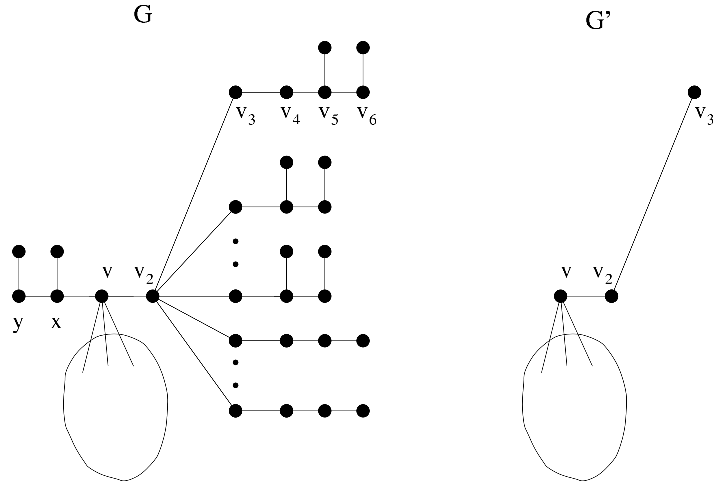


Figure 5: Illustration of $G$ and $G'$ from Lemma 14.

**Proof.** In the proof of this lemma we use the notation from figure 5. First assume that $G$ is TWD and let $D$ be a minimal total dominating set of $G'$. Let $A$ be the vertices at distance 2 and 3 from $v_2$ in $G - vv_2$ that are not leaves. Then $D' := (D \cup A \cup \{v_6, x, y\})\backslash\{v_2\}$ is a minimal total dominating set in $G$. Since $|D'| - |D|$ does not depend on the choice of $D$ the graph $G'$ is TWD if $G$ is TWD.

Next, let $D$ be a minimal total dominating set in $G$. It can be observed that the cardinality of $D\backslash(V(G')\backslash\{v, v_2, v_3\})$ does not depend on the choice of $D$. If $v \in D$ then $v_3 \notin D$. Thus if $D' := (D \cap V(G')) \cup \{v_2\}$ when $v \in D$ and $D' := (D \cap V(G')) \cup \{v_2, v_3\}$ when $v \notin D$ then $|D'| - |D|$ does not depend on $D$. Since $D'$ is a minimal total dominating set in $G'$ the graph $G$ is TWD if $G'$ is TWD. $\square$

**Lemma 15** *Let $G$ be a graph with a non-stem $v \in V(G)$ adjacent to a stem $x$. Assume that a graph from $\mathcal{A}_5$ is attached at $v$. If $H$ is the graph obtained by removing the attached $\mathcal{A}_5$-graph and attaching $P_7$ to $v$ then $G$ is TWD if and only if $H$ is TWD.*

**Proof.** Assume that $G$ and $H$ are as described in the lemma. For both graphs let $P$ denote a path $v_1, v_2, \ldots, v_7$ contained in the graph attached at $v$ such that $vv_1$ is an edge. Let $G'$ be the component of $G - vv_1$ (or $H - vv_1$) containing $v$. Since $G$ contains an admissible set $A$ such that $G' = G - N[A]$ and $H$ contains an admissible set $B$ such that $G' = H - N[B]$ the graph $G'$ is TWD if either $G$ or $H$ is TWD.

Assume $H$ is TWD and let $D$ be a minimal total dominating set in $G$. It can easily be observed that $|D \cap (V(G) \backslash V(G'))|$ does not depend on the choice of $D$. Further $D' := D \cap V(G')$ must be a minimal total dominating set of $G'$. If this is not the case then $v$ is in $D$ and its sole purpose is to dominate $v_1$, so that the sets $D' \cup \{v_3, v_4, v_6, v_7\}$ and $(D' \cup \{v_2, v_3, v_6, v_7\}) \backslash \{v\}$ are both minimal total dominating sets in $H$. But since $H$ is TWD this is a contradiction. Thus it follows that $G$ is TWD if $H$ is TWD. By similar arguments it can be proven that $H$ is TWD if $G$ is TWD. □

**Lemma 16** *Let $G$ be a graph. If a graph from $\mathcal{A}_4$ and a graph from $\mathcal{A}_5$ are attached at a vertex $v \in V(G)$ then $G$ is TWD if and only if the graph $H$ obtained by removing the attached graph from $\mathcal{A}_5$ is TWD.*

**Proof.** First assume that $G$ is TWD. It can be observed that all graphs from $\mathcal{A}_5$ has a total dominating set $D$ such that the attachment vertex is not contained in $D$ and each vertex from $D$ are adjacent to exactly one vertex from $D$. If $D$ is such a set in the graph from $\mathcal{A}_5$ attached at $v$ then $D$ is an admissible set in $G$ and thus $H = G - N[D]$ is TWD.

Now assume that $H$ is TWD and let $D$ be a minimal total dominating set in $G$. Let $G'$ be the graph from $\mathcal{A}_5$ attached at $v$. By considering $G'$ it can be observed that $|D \cap V(G')| = \gamma_t(G')$. Assume first that $D'' := D \cap H$ is not a minimal total dominating set in $H$. Let $v_1, \ldots, v_6$ be a path in the attached graph from $\mathcal{A}_4$ such that $vv_1 \in E(G)$.

If $v \in D$ then $D''$ must dominate $H$ but the only neighbour to $v$ contained in $D$ is the vertex from $N[v] \cap V(G')$. Thus $v$ must be an isolated vertex in $H[D'']$, $v_3 \in D''$ and $(D'' \backslash \{v_3\}) \cup \{v_1\}$ is a minimal total dominating set in $H$. If $v \notin D$ then $v$ is not dominated by $D''$ and $v_3 \in D$. Thus $(D'' \backslash \{v_3\}) \cup \{v_1\}$ is a minimal total dominating set in $H$. In all cases $|D''| = \gamma_t(H)$ and we obtain that $G$ is TWD if $H$ is TWD. □

**Lemma 17** *Let $G$ be a graph with the structure illustrated in figure 6 and assume $N[x]$ does not contain any stems and all vertices at distance two from $x$ in $G - vx$ are adjacent to a stem. Let $H$ be the component of $G - xv$ containing $x$. Then $G$ is TWD if and only if $H$ is TWD and $x$ is special in $H$.*

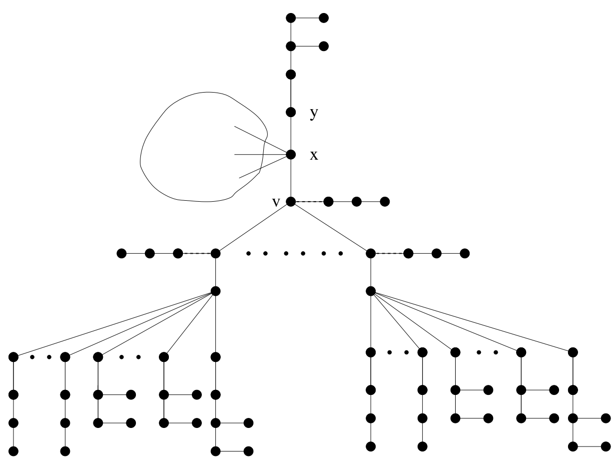


Figure 6: Illustration of $G$.

**Proof.** First assume that $G$ is TWD. Let $G'$ be the union of all components in $G - v$ not containing $x$. By considering $G'$ it can be seen that it has minimal total dominating sets $A$ and $B$ such that $N[A] \cap v \neq \emptyset$, $N[B] \cap v = \emptyset$, $|A| = |B|$ and for $X \in \{A, B\}$ each vertex from $X$ are adjacent to exactly one vertex from $X$. Each of the sets $A$ and $B$ are admissible in $G$. Thus $H = G - N[A]$ and $G - N[B]$ is TWD implying $x$ is special in $H$.

Now assume that $H$ is TWD and let $D$ be a minimal total dominating set of $G$. Let $G'$ be the union of all the graphs from $\mathcal{A}_5$ attached at $v$ (not containing $x$) and let $G'' := G - V(G')$. By considering $G'$ it can be observed that $|D \cap V(G')| = \gamma_t(G')$. In the following we prove that $|D \cap V(G'')|$ does not depend on the choice of $D$. If $v \notin D$ then let $D' := D \cap V(H)$ and otherwise let $D' := ((D \cap V(H))\backslash\{v\}) \cup \{y\}$. Since $N[x]$ does not contain any stems and all vertices at distance two from $x$ in $H$ are adjacent to a stem the set $D'$ must be a minimal total dominating set for $H$ or the graph obtained by attaching a $P_1$ to $x$ in $H$. Since we assume that $x$ is special in $H$ we have $|D'| = \gamma_t(H)$. Thus $|D \cap V(G'')|$ does not depend on the choice of $D$ since $|D \cap V(G'')| = |D'|$ if no $P_3$ is attached at $v$ and $|D \cap V(G'')| = |D'| + 2$ if a $P_3$ is attached at $v$. Thus the graph $G$ is TWD if $H$ is TWD. $\square$

# 3 Main Result

In this section we consider TWD trees that cannot be reduced by any of the decomposition/composition rules. Such a tree is called *reduced* and the following theorem shows that $\{P_2, P_4, P_3 \circ K_1\}$ is the family of reduced trees.

**Theorem 1** *Let $T$ be a TWD tree. Then $T$ is reduced if and only if $T \in \{P_2, P_4, P_3 \circ K_1\}$.*

**Proof.** Assume that $T$ is a reduced TWD and $T \notin \{P_2, P_4, P_3 \circ K_1\}$. In the following we consider a path $P : x_1, \ldots, x_k$ in $T$ such that

1. $x_1$ is a leaf.
2. Any path $x_k, x_{k-1}, u_1, u_2, \ldots, u_l$ in $T$ has length at most $k-1$.
3. If $C$ is the center-vertices in $T$ and $P'$ is a path between $C$ and $x_1$, then $V(P) \subseteq V(C) \cup V(P')$.
4. no path $x_1, \ldots, x_k, x_{k+1}$ satisfy conditions 1-3.

In the following we only say that a graph $H$ with attachment vertex $a$ is attached at $x_i$ if a longest path $x_i, a, u_1, \ldots, u_l$ where $\{a, u_1, \ldots, u_l\} \subseteq V(H)$ has length at most $i-1$ and $a \neq x_{i-1}$.

The only vertex from $P$ that a $P_1$ can be attached at is $x_3$. Since $T$ cannot be reduced it follows from Corollary 1 that a $P_1$ is not attached at $x_i$ for $i \geq 5$, and since $T$ cannot be reduced Lemma 2 implies that a $P_1$ is not attached at $x_2$. If a $P_1$ is attached at $x_4$ then $T \cong P_3 \circ K_1$ by Lemma 6. No vertex of $P$ can have a $P_2$ attached, because it follows from Corollary 1 that a $P_2$ cannot be attached at $x_i$ for $i \geq 5$. Since $T$ is reduced Lemma 3 and Lemma 6 implies that a $P_2$ is not attached at $x_3$ and Lemma 4 implies that a $P_2$ is not attached at $x_4$.

Now consider the vertex $x_4$. Since $T$ is reduced only a $P_3$ or the graph $A_1$ can be attached at $x_4$. By Lemma 4 the graph $A_1$ cannot be attached at $x_4$. If $x_3$ is a stem then it follows from Lemma 4 that a $P_3$ is not attached at $x_4$ and if $x_3$ is not a stem it follows from Lemma 7 that a $P_3$ can not be attached at $x_4$. Thus it can be assumed that $deg(x_4) = 2$.

Consider the graphs that can be attached at $x_5$. Since $T$ is reduced only the graphs $P_3, P_4, A_1$ and $A_2$ can be attached at $x_5$. Since $T$ is reduced Lemma 4 implies that $x_5$ cannot be adjacent to a stem if $x_3$ is a stem, and Lemma 8 implies that $x_5$ cannot be adjacent to a stem if $x_3$ is not a stem. Thus $x_5$ is not adjacent to a stem, and therefore $A_1$ is not attached at $x_5$. If a $P_3 : a, v_1, v_2$ is attached at $x_5$ and $ax_5 \in E(T)$ then $\{v_1, v_2, x_2, x_3, x_4\}$ is contained in a minimal total dominating set $D$ and $D' := (D \backslash \{x_4, v_2\}) \cup \{a\}$ is a total dominating set.

Since $|D'| < |D|$ we obtain a contradiction since $T$ is TWD, so no $P_3$ is attached at $x_5$. Further Lemma 11 implies that $A_2$ is not attached at $x_5$ when $x_3$ is a stem.

Since $T$ is reduced by Lemma 10 the structure near $P$ must be as illustrated in figure 7 when $k \geq 6$.

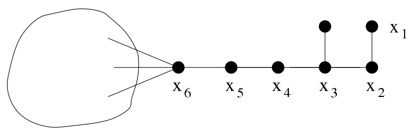


Figure 7: Illustration of structure in $T$ when $k \geq 6$.

Consider the graphs attached at $x_6$. Since $T$ is reduced Lemma 13 implies that $A_1$ is not attached at $x_6$ and from Lemma 12 it follows that $A_3$ is not attached at $x_6$. Thus only $P_3, P_4$ and $A_2$ can be attached at $x_6$. If a $P_3 : a, v_1, v_2$ is attached at $x_6$ and $ax_6 \in E(T)$ then Lemma 4 implies that $x_6$ is not adjacent to a stem and therefor $\{v_1, v_2, x_4, x_5\}$ is contained in a minimal total dominating set $D$. But then $D' := (D \backslash \{x_5, v_2\}) \cup \{a\}$ is a total dominating set and $|D'| < |D|$. Since $T$ is TWD a contradiction is obtained if a $P_3$ is attached at $x_6$. So only $P_4$'s and $A_2$'s can be attached at $x_6$.

Lemma 14 implies that $A_1$ cannot be attached at $x_7$. If a $P_4 : a, v_1, v_2, v_3$ is attached at $x_7$ then Lemma 8 implies $x_7$ is not adjacent to a stem and thus $\{a, v_2, v_3, x_7, x_4\}$ is a subset of a minimal total dominating set $D$. But then $D' := (D \backslash \{a, x_4\}) \cup \{x_6\}$ is a total dominating set that satisfy $|D'| < |D|$. By using similar arguments we obtain that $A_2$ cannot be attached at $x_7$.

Assume that $A_3$ or a graph from $\mathcal{A}_4$ is attached at $x_7$ and let $T'$ denote this subgraph. By considering $T$ it follows that $\{x_5, x_4, x_3, x_2\}$ is contained in a total dominating set $D$. Further $D' := (D \backslash \{x_4, x_5, y\}) \cup \{x_6, x_7\}$ is a total dominating set for a vertex $y \in D \cap V(T')$ at distance two from $x_7$. Since $T$ is TWD it can be assumed that only $P_3$ can be attached at $x_7$. Further Lemma 7 implies that at most one $P_3$ is attached at $x_7$.

Consider any graph $G'$ obtained by attaching a graph from $\mathcal{A}_5$ to the center of a star $K_{1,t}$ for some $t \geq 0$. Since $G'$ is not TWD it follows there must be a component $C$ of $T - x_8$, not containing the vertex $x_1$, such that $C$ does not contain a admissible set $D$ that satisfies $y \notin D$ and either $y \in N[D]$ or $N(y) \cap C \subseteq N[D]$ where $y$ is the vertex from $C$ adjacent to $x_8$. If this is not the case then the union of such admissible sets in all components of $T - x_8$ not containing $x_1$ would be an admissible set $D$ such that $T - N[D]$ had a component isomorphic to a graph like $G'$. Since $T$ is TWD it follows from Observation 1 and Lemma 1 that this is a contradiction.

Consider a path $y, v_1, v_2$ in $C$ such that $yx_8 \in E(T)$. Since $\{v_1, v_2\}$ can not be an admissible $G - [\{v_1, v_2\}]$ must have a isolated vertex. Since $T$ is reduced Lemma 15 implies $y$ is not a stem and therefore $v_1$ or $v_2$ must be adjacent to a stem in $C - \{y, v_1, v_2\}$. If $v_1$ is adjacent to such a stem $u$ then since $T$ is reduced Corollary 1 shows that $u$ must be adjacent to a stem $z$ but then $\{v_1, u, z\}$ is contained in an admissible set in $C \backslash \{y\}$ which is a contradiction to the choice of $C$. Thus $v_2$ must be adjacent to a stem $u \neq v_1$.

Thus Corollary 1 and Lemma 7 imply that $T$ must have the structure illustrated in figure 8. Let $A$ be as illustrated in figure 8. Further let $D$ be all vertices from $V(C)\backslash\{y\}$ at distance at least three from $A$. Now $A \cup D$ is a total dominating set of $C - y$ and thus there is a minimal total dominating set $D'$ of $C - y$ such that $D' \subseteq A \cup D$. If no $A_3$-graph is attached at $y$ then $D'$ is an admissible set in $T$ contradicting the choice of $C$. Thus we may assume that a $A_3$ is attached at $y$.

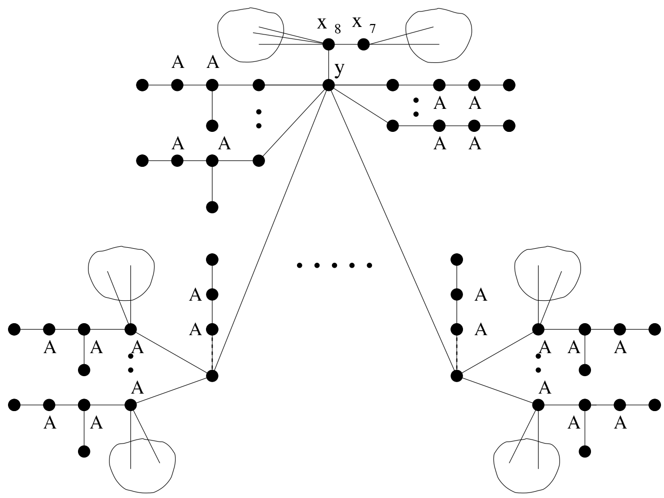

Figure 8: Illustration of $T$.

Now consider the graphs attached at $x_8$. Only $P_3, P_4, A_1, A_2, A_3$ or a graph from $\mathcal{A}_4 \cup \mathcal{A}_5$ can be attached at $x_8$. By Lemma 16 a graph from $\mathcal{A}_4$ is not attached at $x_8$ and by Lemma 15 $A_1$ can not be attached at $x_8$. If one of the graphs $P_4, A_2, A_3$ is attached at $x_8$ and $a$ denotes the attachment-vertex from such a graph, then since $x_8$ is not adjacent to a stem then $\{a, x_2, x_3, x_4, x_5, x_8\}$ is contained in a minimal total dominating set $D$. Further $D' := (D\backslash\{a, x_5\}) \cup \{x_7\}$

is a total dominating set. Since $T$ is TWD none of these graphs can be attached at $x_8$. Thus $T$ has the structure as the graph from Lemma 17 and since $T$ is reduced this is a contradiction.

Now let $P$ be a subpath of a diametrical path in $T$. By the above arguments $k \leq 7$ and $k \geq 3$ since $T \ncong P_2$. Thus there must be a graph attached at $x_k$ and the information about graphs attached at $x_3, x_4, x_5, x_6$ and $x_7$ implies that $T \in \mathcal{A}_4$.

This proves the statement since all the graphs from $\{P_2, P_4, P_3 \circ K_1\}$ are reduced graphs and Lemma 4, Lemma 10, Lemma 11 and Lemma 14 imply that no graph from $\mathcal{A}_4$ is reduced. $\square$

# References


[1] T.W. Haynes, S.T. Hedetniemi, and P.J. Slater, *Fundamentals of Domination in Graphs.* Marcel Dekker, New York, 1998.

[2] T.W. Haynes, S.T. Hedetniemi, and P.J. Slater (eds.), *Domination in Graphs: Advanced Topics.* Marcel Dekker, New York, 1998.

[3] O. Ore, *Theory of graphs. Amer. Math. Soc. Transl.* **38** (Amer. Math. Soc., Providence, RI, 1962), 206–212.

# Total Domination in Partitioned Trees and Partitioned Graphs with Minimum Degree Two

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**Abstract**

Let $G = (V, E)$ be a graph and let $S \subseteq V$. A set of vertices in $G$ totally dominates $S$ if every vertex in $S$ is adjacent to some vertex of that set. The least number of vertices needed in $G$ to totally dominate $S$ is denoted by $\gamma_t(G, S)$. When $S = V$, $\gamma_t(G, V)$ is the well studied total domination number $\gamma_t(G)$. We wish to maximize the sum $\gamma_t(G) + \gamma_t(G, V_1) + \gamma_t(G, V_2)$ over all possible partitions $V_1, V_2$ of $V$. We call this maximum sum $f_t(G)$. For a graph $H$, we denote by $H \circ P_2$ the graph obtained from $H$ by attaching a path of length 2 to each vertex of $H$ so that the resulting paths are vertex-disjoint. We show that if $G$ is a tree of order $n \geq 4$ and $G \notin \{P_5, P_6, P_7, P_{10}, P_{14}\}$, then $f_t(G) \leq 14n/9$ with equality if and only if $G \in \{P_9, P_{18}\}$ or $G = (T \circ P_2) \circ P_2$ for some tree $T$. If $G$ is a connected graph of order $n$ with minimum degree at least two, we establish that $f_t(G) \leq 3n/2$ with equality if and only if $G$ is a cycle of order congruent to zero modulo 4.

**Keywords:** partitioned graphs; total domination
**AMS subject classification: 05C69**

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# 1 Introduction

In this paper, we continue the study of the concept of partitions and domination in graphs introduced by Hartnell and Vestergaard [5], and studied, for example, in [7, 8, 9]. Here we study partitions and total domination in graphs. Throughout this article, only undirected simple graphs without loops or multiple edges are considered.

For notation and graph theory terminology we in general follow [1, 3]. Specifically, let $G = (V, E)$ be a graph with vertex set $V$ of order $n = |V|$ and edge set $E$ of size $m = |E|$, and with no isolated vertices. For sets $S, T \subseteq V$, $S$ *totally dominates* $T$ if every vertex in $T$ is adjacent to some vertex of $S$. If $S$ totally dominates $V$, then $S$ is called a *total dominating set*, denoted TDS, of $G$. Every graph without isolated vertices has a TDS, since $S = V$ is such a set. The *total domination number* of $G$, denoted by $\gamma_t(G)$, is the minimum cardinality of a TDS. For $U \subseteq V$, we let $\gamma_t(G, U)$ denote the minimum cardinality of a set of vertices in $G$ that totally dominates $U$. Hence, $\gamma_t(G, V) = \gamma_t(G)$. If $U = \emptyset$, we define $\gamma_t(G, U) = 0$. A set of cardinality $\gamma_t(G, U)$ that totally dominates $U$ in $G$ we call a $\gamma_t(G, U)$-set. If $U = V$, we also call a $\gamma_t(G, U)$-set a $\gamma_t(G)$-set. Total domination in graphs was introduced by Cockayne, Dawes, and Hedetniemi [2] and is now well studied in graph theory. The literature on this subject has been surveyed and detailed in the two books by Haynes, Hedetniemi, and Slater [3, 4].

By a *partition* of the vertices of a graph $G = (V, E)$, we mean two subsets $V_1, V_2$ of $V$ with $V = V_1 \cup V_2$ and $V_1 \cap V_2 = \emptyset$; $\{V_1, V_2\} = \{\emptyset, V\}$ is permitted. Given a partition $\mathcal{P} = \{V_1, V_2\}$ of $V$, we define the *label* of a vertex $v$ in $\mathcal{P}$, denoted $\ell_{\mathcal{P}}(v)$, as the number $i \in \{1, 2\}$ such that $v \in V_i$. For a graph $G$, and a partition $V_1, V_2$ of $V$, we define $g_t(G; V_1, V_2)$ and $f_t(G; V_1, V_2)$ by

$$\begin{aligned} g_t(G; V_1, V_2) &= \gamma_t(G, V_1) + \gamma_t(G, V_2), \\ f_t(G; V_1, V_2) &= \gamma_t(G) + g_t(G; V_1, V_2), \end{aligned}$$

and $g_t(G)$ and $f_t(G)$ by

$$\begin{aligned} g_t(G) &= \max\{g_t(G; V_1, V_2) \mid V_1, V_2 \text{ is a partition of } V\}, \\ f_t(G) &= \max\{f_t(G; V_1, V_2) \mid V_1, V_2 \text{ is a partition of } V\}. \end{aligned}$$

Our aim in this paper is twofold. We wish to establish a sharp upper bound for the function $f_t(G)$ in terms of the order $n$ of a graph $G$ in two cases. Firstly we establish an upper bound for $f_t(G)$ in the case when $G$ is a tree of order at least 4. Secondly we establish an upper bound for $f_t(G)$ in the case when $G$ is a connected graph with minimum degree at least two. In both cases we characterize the graphs achieving equality in these bounds.

A motivation for studying total domination in partitioned graphs is that of minimizing the number of file servers needed to serve workstations in a computer network. Some files (such as data and text files) are compatible with all computers on the network while other files (such as binary code files) are only compatible with a particular type of workstation (Mac, Sun, Sparc, for example). The file server ordinarily is shared among the workstations and is located one communication step away from the workstation it caters to. Such a workstation is said to be totally dominated by the file server. The computer network can be modelled by a graph $G$ where the vertices represent the workstations and the file servers and where the edges represent the connections between the workstations and their local file servers. The requirement that the file server is adjacent to the workstation it serves corresponds in the associated graph to demanding that some collection of text file servers totally dominate every vertex of the graph, since each workstation must have access to common requirements of data, text, latex, emacs files, Internet connection, open windows, dos, etc. We are interested in workstations of two types, type 1 and 2, and for each $i = 1, 2$, some collection of file servers must totally dominate a fixed set $V_i$ of all vertices of the graph of type $i$. We wish to minimize the number of file servers; that is, we wish to minimize the sum $\gamma_t(G) + \gamma_t(G, V_1) + \gamma_t(G, V_2)$ where $\gamma_t(G)$ is the number of text file servers needed so that every workstation has access to data, text, internet connection, latex, emacs files, and $\gamma_t(G, V_i)$ is the number of specialized file servers needed for workstations of type $i$. We wish to determine the maximum of the minimum sums $\gamma_t(G) + \gamma_t(G, V_1) + \gamma_t(G, V_2)$ over all possible partitions $V_1$, $V_2$ of the two types of workstations; that is, we wish to determine the maximum number of text file servers and specialized file servers needed to serve the workstations over all possible partitions of the workstations into the two given types.

### 1.1 Notation

Let $G = (V, E)$ be a graph and let $v \in V$ and $S \subseteq V$. The *open neighborhood* of $v$ in $G$ is $N(v) = \{u \in V \mid uv \in E\}$, while the *open neighborhood* of $S$ is the set $N(S) = \cup_{v \in S} N(v)$. Hence for a set $U \subseteq V$, the set $S$ *totally dominates* $U$ if $U \subseteq N(S)$. For a set $S \subseteq V$, the subgraph induced by $S$ is denoted by $G[S]$. A vertex of degree $k$ we call a *degree-$k$ vertex*. A degree-1 vertex we call a *leaf* (or an end-vertex), and a vertex adjacent to a leaf we call a *support vertex*. The minimum (resp., maximum) degree among the vertices of $G$ is denoted by $\delta(G)$ (resp., $\Delta(G)$). For disjoint subsets $S$ and $T$ of vertices, we denote by $[S, T]$ the set of edges of $G$ with one end in $S$ and the other in $T$.

A subset $S$ of vertices in a graph $G$ is an *open packing* if the open neighborhoods of vertices in $S$ are pairwise disjoint, i.e., no two vertices from $S$ have a common neighbor, but they may be adjacent.

A set $M$ of edges of $G$ is a *matching* if no two edges in $M$ are incident to the same vertex. A *perfect matching* in $G$ is a matching with the property that every vertex is incident with an edge of the matching.

A cycle on $n \geq 3$ vertices is denoted by $C_n$ and a path on $n \geq 1$ vertices by $P_n$. A path $P_1$ is called a trivial path. For $r \geq 3$ and $s \geq 1$, we denote by $L_{r,s}$ the graph obtained by joining with an edge a vertex in $C_r$ to an end-vertex of $P_s$. We call the graph $L_{r,s}$ a *key*.

For a graph $H$, we denote by $H \circ P_2$ the graph of order $3|V(H)|$ obtained from $H$ by attaching a path of length 2 to each vertex of $H$ so that the resulting paths are vertex-disjoint. The graph $H \circ P_2$ is also called the 2-*corona* of $H$.

## 2 Known Results

In this section, we mention the previous best known upper bounds for $f_t(G)$ when $G$ is a tree of order at least 3 and when $G$ is a connected graph with minimum degree at least two.

Let $G = (V, E)$ be a graph and let $S \subseteq V$. Every minimum TDS in $G$ totally dominates the set $S$. Hence, $\gamma_t(G, S) \leq \gamma_t(G)$. This implies that $f_t(G) \leq 3\gamma_t(G)$. When $G$ is a tree of order $n \geq 3$, then Cockayne, Dawes, and Hedetniemi [2] showed that $\gamma_t(G) \leq 2n/3$. When $G$ is a connected graph of order $n$ with $\delta(G) \geq 2$, and $G \notin \{C_3, C_5, C_6, C_{10}\}$, then it is shown in [6] that $\gamma_t(G) \leq 4n/7$. Hence the following two results are immediate consequences of known upper bounds on the total domination number of a graph.

**Fact 1** ([2]) *If $T$ is a tree of order $n \geq 3$, then $f_t(G) \leq 2n$.*

**Fact 2** ([6]) *If $G \notin \{C_3, C_5, C_6, C_{10}\}$ is a connected graph of order $n$ with $\delta(G) \geq 2$, then $f_t(G) \leq 12n/7$.*

## 3 Main Results

We shall prove:

**Theorem 1** *If $T$ is a tree of order $n \geq 4$ and $T \notin \{P_5, P_6, P_7, P_{10}, P_{14}\}$, then*

$f_t(T) \le 14n/9$ *with equality if and only if* $T \in \{P_9, P_{18}\}$ *or* $T = (T' \circ P_2) \circ P_2$ *for some tree* $T'$.

The tree $(K_1 \circ P_2) \circ P_2$, for example, is shown in Figure 1.

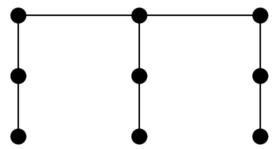

Figure 1: The tree $(K_1 \circ P_2) \circ P_2$.

**Theorem 2** *If* $G$ *is a connected graph of order* $n$ *with* $\delta(G) \ge 2$, *then* $f_t(G) \le 3n/2$ *with equality if and only if* $G \cong C_n$ *where* $n \equiv 0 \,(mod\, 4)$.

# 4 Proof of Theorem 1

## 4.1 Preliminary Results

The total domination number of a cycle $C_n$ or a path $P_n$ on $n \ge 3$ vertices is easy to compute.

**Lemma 1** ([6]) *For* $n \ge 3$, $\gamma_t(P_n) = \gamma_t(C_n) = \lfloor n/2 \rfloor + \lceil n/4 \rceil - \lfloor n/4 \rfloor$.

Thus for $G \in \{P_n, C_n\}$, if $n \ge 3$ is odd, then $\gamma_t(G) = (n+1)/2$ and if $n$ is congruent to zero modulo 4, then $\gamma_t(G) = n/2$. Finally if $n$ is congruent to two modulo 4, then $\gamma_t(G) = (n+2)/2$.

The total domination number of a key $L_{r,s}$ of order (and size) $r+s$ was determined in [6]. As a consequence of this result, we have the following upper bound on $\gamma_t(L_{r,s})$.

**Lemma 2** ([6]) *For* $r \ge 3$ *and* $s \ge 1$, *if* $G$ *is a key* $L_{r,s}$ *of order* $n = r + s$, *then* $\gamma_t(G) \le (n+2)/2$ *with equality if and only if* $r \equiv 2 \,(mod\, 4)$ *and* $s \equiv 0 \,(mod\, 4)$.

The following lemmas follow immediately from the definitions of $f_t(G)$ and $g_t(G)$.

**Lemma 3** *If $G'$ is a spanning subgraph of a graph $G$ with $\delta(G') \geq 1$, then $g_t(G) \leq g_t(G')$.*

**Lemma 4** *If $G$ is a graph with no isolated vertex, then $f_t(G) = \gamma_t(G) + g_t(G)$.*

We shall use the obvious observation that for a graph $G$ with induced subgraphs $G_1, G_2$ having no isolated vertices and satisfying $V(G) = V(G_1) \cup V(G_2)$, we have that

$$\begin{aligned} \gamma_t(G) &\leq \gamma_t(G_1) + \gamma_t(G_2), \\ g_t(G) &\leq g_t(G_1) + g_t(G_2), \\ f_t(G) &\leq f_t(G_1) + f_t(G_2). \end{aligned}$$

The following lemma follows readily from the definition of an open packing.

**Lemma 5** *Let $G = (V, E)$ be a path $v_1, v_2, \ldots, v_n$ of order $n$, and let $V_1$, $V_2$ be a partition of $V$. If both $V_1$ and $V_2$ are open packings in $G$, then the labels of $V(P_n)$ come in alternating pairs but the beginning and the end may be a pair or a single label. More precisely, renaming the sets $V_1$ and $V_2$ if necessary, we have*

$$V_1 = \left( \bigcup_{i=0}^{\lfloor (n-1)/4 \rfloor} \{v_{4i+1}\} \right) \cup \left( \bigcup_{i=0}^{\lfloor (n-2)/4 \rfloor} \{v_{4i+2}\} \right)$$

*or*

$$V_1 = \left( \bigcup_{i=0}^{\lfloor (n-1)/4 \rfloor} \{v_{4i+1}\} \right) \cup \left( \bigcup_{i=0}^{\lfloor (n-4)/4 \rfloor} \{v_{4(i+1)}\} \right),$$

*with the remaining vertices in $V_2$.*

**Definition 1** *For a graph $G = (V, E)$, we define a partition $V_1$, $V_2$ of $V$ to be a* good partition *if both $V_1$ and $V_2$ are open packings in $G$.*

The following lemmas will prove to be useful when proving our main results.

**Lemma 6** *Let $G = (V, E)$ be a graph of order $n \geq 2$ with no isolated vertices, and let $V_1$, $V_2$ be a partition of $V$. Then, $V_1$, $V_2$ is a good partition of $V$ if and only if $\gamma_t(G, V_1) + \gamma_t(G, V_2) = n$.*

**Proof.** Suppose that $V_1$, $V_2$ is a good partition of $V$. Then for $i \in \{1,2\}$, no two vertices from $V_i$ can be dominated by a common vertex, and so $\gamma_t(G,V_1)+\gamma_t(G,V_2) = |V_1|+|V_2| = n$. This establishes the necessity. To prove the sufficiency, suppose that $V_1$, $V_2$ is not a good partition of $V$. We may assume that $V_1$ is not an open packing in $G$. Thus there exist two vertices in $V_1$ that have a common neighbor, implying that $\gamma_t(G,V_1) \le |V_1|-1$. Hence since $\gamma_t(G,V_2) \le |V_2|$, we have that $\gamma_t(G,V_1)+\gamma_t(G,V_2) \le n-1$. □

**Lemma 7** *For $n \ge 2$, $g_t(P_n) = n$ and $f_t(P_n) = \lfloor 3n/2 \rfloor + \lceil n/4 \rceil - \lfloor n/4 \rfloor$.*

**Proof.** Since every path has a good partition of its vertex set, we have by Lemma 6 that $g_t(P_n) = n$. The desired result now follows from Lemmas 1 and 4. □

Thus by Lemma 7, if $n \ge 3$ is odd, then $f_t(P_n) = (3n+1)/2$; if $n \equiv 0 \pmod 4$, then $f_t(P_n) = 3n/2$; if $n \equiv 2 \pmod 4$, then $f_t(P_n) = (3n+2)/2$.

**Lemma 8** *If $G = (V,E)$ is a path of order $n \ge 2$, and $V_1$, $V_2$ is not a good partition of $V$, then $f_t(G;V_1,V_2) \le 3n/2$ with strict inequality if $n \not\equiv 2 \pmod 4$.*

**Proof.** By Lemma 6, $\gamma_t(G,V_1)+\gamma_t(G,V_2) \le n-1$. By Lemma 1, $\gamma_t(G) \le (n+2)/2$ with strict inequality if $n \not\equiv 2 \pmod 4$. Hence, $f_t(G;V_1,V_2) \le 3n/2$ with strict inequality if $n \not\equiv 2 \pmod 4$. □

The following lemma is an immediate consequence of Lemma 8.

**Lemma 9** *If $G = (V,E)$ is a path of order $n \ge 2$, and $V_1$, $V_2$ is a partition of $V$ for which $f_t(G;V_1,V_2) > 3n/2$, then $V_1$, $V_2$ is a good partition of $V$.*

**Lemma 10** *If $G$ is a graph of order $n$ without isolated vertices and $S \subseteq V(G)$, then $g_t(G) \le n+2|S|-|N(S)|$.*

**Proof.** Let $G = (V,E)$ and let $V_1$, $V_2$ be a partition of $V$. Let $i \in \{1,2\}$. For each vertex $v \in V_i \backslash N(S)$, we choose an adjacent vertex and call the resulting set of such vertices $S_i'$. Then, $S \cup S_i'$ totally dominates $V_i$ in $G$, and so $\gamma_t(G,V_i) \le |S|+|S_i'|$. Thus, $g_t(G;V_1,V_2) \le 2|S|+|S_1'|+|S_2'| \le 2|S|+|V \setminus N(S)| = n+2|S|-|N(S)|$. Thus for every partition $V_1$, $V_2$ of $V$, $g_t(G;V_1,V_2) \le n+2|S|-|N(S)|$. Therefore, $g_t(G) \le n+2|S|-|N(S)|$. □

As a special case of Lemma 10, we have the following result.

**Lemma 11** *If $G$ is a graph of order $n$ with no isolated vertex and maximum degree at least 3, then $g_t(G) \leq n - 1$.*

**Proof.** Let $v$ be a vertex of maximum degree at least 3 and let $S = \{v\}$. Then, $|S| = 1$ and $|N(S)| \geq 3$, and so the desired result follows from Lemma 10. □

**Lemma 12** *If $T$ is a graph of order $n$ that can be obtained from a star on at least four vertices by subdividing some (including the possibility of none) of the edges exactly once, then $f_t(T) < 3n/2$.*

**Proof.** For integers $r \geq k \geq 0$ with $r \geq 3$, let $T = (V, E)$ be obtained from a star $K_{1,r}$ by subdividing $k$ edges exactly once. If $k = 0$, then $n = r + 1 \geq 4$ and $f_t(T) \leq 5 < 3n/2$. Hence we may assume that $k \geq 1$. Then, $\gamma_t(T) = k+1$. Let $V_1$, $V_2$ be a partition of $V$. Then, $\gamma_t(T, V_1) + \gamma_t(T, V_2) \leq k + 3$, and so $f_t(T; V_1, V_2) \leq 2k + 4$. Since $r \geq k$ and $r \geq 3$, we have $3n/2 = 3(k + r + 1)/2 = (3k + r)/2 + r + 3/2 \geq 2k + 9/2$. Thus for every partition $V_1$, $V_2$ of $V$, $f_t(T; V_1, V_2) < 3n/2$. Therefore, $f_t(T) < 3n/2$. □

Next we define a special set $\mathcal{S}$ of small paths.

**Definition 2** *Let $\mathcal{S} = \{P_1, P_2, P_3, P_5, P_6, P_7, P_{10}, P_{14}\}$.*

As a consequence of the remark after Lemma 7 we have the following result.

**Lemma 13** *If $T \in \mathcal{S}$ has order $n \geq 2$, then $f_t(T) = (3n + 1)/2$ if $n$ is odd; otherwise, $f_t(T) = (3n + 2)/2$.*

A proof of the following lemma is a simple exercise and is omitted.

**Lemma 14** *Let $T = (V, E)$ be a path in $\mathcal{S}$. If $|V| \geq 2$ and $v \in V$ is neither a leaf of a $P_5$ nor a center of a $P_7$, then there exists a $\gamma_t(T)$-set containing $v$.*

**Definition 3** *Let $\mathcal{T} = \{T \mid T = (T' \circ P_2) \circ P_2$ for some tree $T'\}$.*

## 4.2 Proof of Theorem 1

Recall Theorem 1.

**Theorem 1** *If $T \notin \mathcal{S}$ is a tree of order $n \geq 4$, then $f_t(T) \leq 14n/9$ with equality if and only if $T \in \{P_9, P_{18}\}$ or $T \in \mathcal{T}$.*

**Proof.** We proceed by induction on $n$. When $n = 4$, either $T = K_{1,3}$, in which case $f_t(T) = 5$, or $T = P_4$, in which case $f_t(T) = 6$. In both cases, $f_t(T) < 14n/9$. This establishes the base case. For the inductive hypothesis, let $n \geq 5$ and assume that for all trees $T' \notin \mathcal{S}$ of order $n'$, where $4 \leq n' < n$, $f_t(T') \leq 14n'/9$ with equality if and only if $T' \in \{P_9, P_{18}\}$ or $T' \in \mathcal{T}$.

So let $T = (V, E)$ be a tree of order $n$ with $T \notin \mathcal{S}$. The following observation follows from Lemma 1.

**Observation 1** *If $T = P_n$, then $f_t(T) \leq 14n/9$ with equality if and only if $T \in \{P_9, P_{18}\}$.*

By Observation 1, we may assume that $T$ is not a path, for otherwise the desired result follows. With this assumption, we have the following observation by Lemma 11.

**Observation 2** $g_t(T) \leq n - 1$.

**Observation 3** *If $T$ contains a path on five vertices with one end a leaf in $T$ and with each internal vertex a degree-2 vertex in $T$, then $f_t(T) < 14n/9$.*

**Proof.** Let $P\colon v, v_1, v_2, v_3, v_4$ be a path in $T$ where $\deg_T(v_4) = 1$ and $\deg_T(v_i) = 2$ for $i = 1, 2, 3$. Let $T_1$ and $T_2$ be the components of $T - vv_1$ containing $v$ and $v_1$, respectively. Then, $T_1$ is a tree of order $n_1 = n - 4$, while $T_2 = P_4$, and so $g_t(T_2) = n_2 = 4$ and $f_t(T_2) = 6$. Since $T$ is not a path, $n_1 \geq 3$.

Suppose $T_1$ is a path. Then, $g_t(T_1) = n_1$ and, by Lemma 1, $f_t(T_1) \leq (3n_1 + 2)/2$. Thus, $g_t(T_1) + g_t(T_2) = n$. By Observation 2, $g_t(T) \leq n - 1$, and so $g_t(T) \leq g_t(T_1) + g_t(T_2) - 1$. Thus, by Lemmas 3 and 4, $f_t(T) = \gamma_t(T) + g_t(T) \leq \gamma_t(T_1) + \gamma_t(T_2) + g_t(T_1) + g_t(T_2) - 1 = f_t(T_1) + f_t(T_2) - 1 \leq (3n_1 + 2)/2 + 6 - 1 = 3n/2 < 14n/9$. Hence we may assume that $T_1$ is not a path. In particular, $T_1 \notin \mathcal{S}$ and $n_1 \geq 4$. Thus, by the inductive hypothesis, $f_t(T) \leq f_t(T_1) + f_t(T_2) \leq 14n_1/9 + 6 < 14n/9$. □

By Observation 3, we may assume that $T$ contains no path on five vertices with one end a leaf in $T$ and with each internal vertex a degree-2 vertex in $T$.

Let $V_1$, $V_2$ be a partition of $V$. For each edge $uv \in E$, let $T_u$ and $T_v$ denote the components of $T - uv$ containing $u$ and $v$, respectively. If $T_u \in \mathcal{S}$, then we orient the edge from $u$ to $v$, while if $T_v \in \mathcal{S}$, then we orient the edge from $v$ to $u$. (Possibly an edge may be oriented in both directions.)

**Observation 4** *If an edge of $T$ has no orientation, then $f_t(T) \le 14n/9$ with equality if and only if $T \in \mathcal{T}$.*

**Proof.** Suppose that an edge $uv \in E$ has no orientation. Applying the inductive hypothesis to $T_u$ and $T_v$, we have that for $x \in \{u, v\}$, $f_t(T_x) \le 14|V(T_x)|/9$ with equality if and only if $T_x \in \{P_9, P_{18}\}$ or $T_x \in \mathcal{T}$. Hence, $f_t(T) \le f_t(T_u) + f_t(T_v) \le 14|V(T_u)|/9 + 14|V(T_v)|/9 = 14n/9$. Thus if $f_t(T_x) < 14|V(T_x)|/9$ for some $x \in \{u, v\}$, then $f_t(T) < 14n/9$. Suppose then that for $x \in \{u, v\}$, $f_t(T_x) = 14|V(T_x)|/9$, and so $T_x \in \{P_9, P_{18}\}$ or $T_x \in \mathcal{T}$.

Suppose that one of $T_u$ and $T_v$, say $T_u$, is a path. Then, $T_u \in \{P_9, P_{18}\}$ and at least one leaf in $T_u$ is a leaf in $T$ that is the end of a path on five vertices every internal vertex of which has degree 2 in $T$, contrary to assumption.

Hence both $T_u$ and $T_v$ are in the family $\mathcal{T}$. Let $G \cong (P_1 \circ P_2) \circ P_2$. Then both $T_u$ and $T_v$ have disjoint copies of $G$ as a spanning subgraph. Thus, $T$ has as a spanning subgraph the graph $H = kG$, consisting of $k$ disjoint copies of $G$, for some integer $k \ge 2$, where $u$ and $v$ belong to different copies of $G$ in $H$. Hence, $n = 9k$. Let $G_u$ and $G_v$ be the copies of $G$ in $H$ that contain $u$ and $v$, respectively. Let $T_{uv} = G_u \cup G_v \cup \{uv\}$.

We proceed further with two observations about the graph $G$. We observe first that $\gamma_t(G) = 6$, while $g_t(G) = |V(G)| - 1 = 8$, and so $f_t(G) = 14 = 14|V(G)|/9$. We observe secondly that for every vertex of $G$ there exists a $\gamma_t(G)$-set containing it and if $w$ is a leaf in $G$ or a support vertex in $G$, then $\gamma_t(G, V(G) \setminus \{w\}) = \gamma_t(G) - 1$.

Suppose that $u$ is a leaf or a support vertex in $G_u$. Then it follows from our two earlier observations about the graph $G$ that $\gamma_t(T_{uv}) \le \gamma_t(G_u) + \gamma_t(G_v) - 1$, implying that $\gamma_t(T) \le k\gamma_t(G) - 1 = 6k - 1$. Thus since $g_t(T) \le kg_t(G) = 8k$, we have that $f_t(T) \le 14k - 1 = 14n/9 - 1$. Hence we may assume that $u$ is neither a leaf nor a support vertex in $G_u$. Similarly, $v$ is neither a leaf nor a support vertex in $G_v$.

Suppose that $u$ or $v$ is the vertex of degree-3 in $G_u$ or $G_v$, respectively. Then applying Lemma 10 to the tree $T_{uv}$ with $S = \{u, v\}$ we have that $g_t(T_{uv}) \le$

$|V(G_u)|+|V(G_v)|+2|S|-|N(S)| \le 18+4-7 = 15$. Thus, $g_t(T) \le g_t(T_{uv})+(k-2)g_t(G) \le 8k-1$ while $\gamma_t(T) \le k\gamma_t(G) = 6k$, and so $f_t(T) \le 14k-1 = 14n/9-1$. Hence we may assume that neither $u$ nor $v$ is the vertex of degree 3 in $G_u$ or $G_v$, respectively.

If $k = 2$, then $T = (T' \circ P_2) \circ P_2$ where $T' = P_2$ consists of the vertices $u$ and $v$, whence $T \in \mathcal{T}$. Hence we may assume that $k \ge 3$.

Assume that $F \cup (k-3)G$ is a spanning subgraph of $T$ where $F = P_9 \circ P_2$. Let $v_1, v_2, \ldots, v_9$ be the vertices from the path $P_9$ in $F$. Then applying Lemma 10 to the graph $F$ with $S = \{v_2, v_3, v_6, v_7\}$ we obtain $g_t(F) \le 27 + 8 - 12 = 23$. Thus, $g_t(T) \le g_t(F) + (k-3)g_t(G) \le 8k-1$ while $\gamma_t(T) \le k\gamma_t(G) = 6k$, and so $f_t(T) \le 14k-1 = 14n/9-1$. Hence we may assume that $(P_9 \circ P_2) \cup (k-3)G$ is not a spanning subgraph of $T$. It follows that the degree of every vertex in $G_u \cup G_v$, different from $u$ and $v$, is unchanged in $T$. Thus for $x \in \{u, v\}$, if $T_x = (T'_x \circ P_2) \circ P_2$ for some tree $T'_x$, then we have that $u \in V(T'_u)$ and $v \in V(T'_v)$. This implies that $T = (T' \circ P_2) \circ P_2$ where $T'$ is the tree $T'_u \cup T'_v \cup \{uv\}$. Thus, $T \in \mathcal{T}$. Hence we have established that either $f_t(T) < 14n/9$ or $f_t(T) = 14n/9$ and $T \in \mathcal{T}$. □

**Observation 5** *If an edge of $T$ is oriented in both directions, then $f_t(T) \le 14n/9$ with equality if and only if $T = (P_1 \circ P_2) \circ P_2$.*

**Proof.** Suppose that an edge $uv \in E$ is oriented in both directions. Hence both components $T_u$ and $T_v$ of $T - uv$ are contained in $\mathcal{S}$. Since both $T_u$ and $T_v$ are paths, $g_t(T_u) + g_t(T_v) = n$. By Observation 2, $g_t(T) \le n-1$, and so $g_t(T) \le g_t(T_u) + g_t(T_v) - 1$.

Since $T$ is not a path, $\deg_T(u) \ge 3$ or $\deg_T(v) \ge 3$. If both $\deg_T(u) \ge 3$ and $\deg_T(v) \ge 3$, then applying Lemma 10 to the tree $T$ with $S = \{u, v\}$, we have $g_t(T) \le n-2 = g_t(T_u)+g_t(T_v)-2$. Thus since $\gamma_t(T) \le \gamma_t(T_u)+\gamma_t(T_v)$, we have by Lemma 13 that $f_t(T) \le f_t(T_u) + f_t(T_v) - 2 \le (3|V(T_u)|+2)/2 + (3|V(T_u)|+2)/2 - 2 = 3n/2 < 14n/9$.

Hence we may assume that either $\deg_T(u) \ge 3$ or $\deg_T(v) \ge 3$, but not both. We may assume that $\deg_T(u) \ge 3$, and so $\deg_T(v) \le 2$. By our assumption following Observation 3, we have that $T_v \in \{P_1, P_2, P_3\}$.

Suppose $T_v = P_1$, and so $|V(T_u)| = n-1$. If there is a $\gamma_t(T_u)$-set containing $u$, then $\gamma_t(T) \le \gamma_t(T_u)$, implying that $f_t(T) \le \gamma_t(T_u) + g_t(T) \le (|V(T_u)| + 2)/2 + n - 1 = (3n-1)/2 < 14n/9$. On the other hand, if there is no $\gamma_t(T_u)$-set containing $u$, then, by Lemma 14, $T_u = P_7$ and $u$ is the central vertex of this $P_7$. But then $n = 8$, $\gamma_t(T) = 5$ and $g_t(T) \le n-1 = 7$, implying that $f_t(T) \le 12 = 3n/2 < 14n/9$. Hence we may assume that $T_v \in \{P_2, P_3\}$.

As observed earlier, $g_t(T) \leq g_t(T_u) + g_t(T_v) - 1$. Thus, $f_t(T) \leq f_t(T_u) + f_t(T_v) - 1$. Hence, by Lemma 13, $f_t(T) \leq (3n + \ell)/2$ where $\ell$ denotes the number of even components of $T - uv$. If $\ell = 0$, then $f_t(T) \leq 3n/2 < 14n/9$, as desired. Hence we may assume that $\ell \in \{1, 2\}$.

Suppose that $\ell = 1$, and so $f_t(T) \leq (3n+1)/2$. If $n > 9$, then $f_t(T) < 14n/9$. Hence we may assume that $n \leq 9$. Suppose firstly that $T_v = P_2$ and $T_u$ is of odd order. If $T_u \neq P_7$ or if $T_u = P_7$ but $u$ is not the central vertex of $T_u$, then there is a $\gamma_t(T_u)$-set containing $u$, and so $\gamma_t(T) \leq \gamma_t(T_u) + 1$, implying that $f_t(T) \leq \gamma_t(T_u) + 1 + g_t(T) \leq (|V(T_u)|+1)/2 + 1 + n - 1 < 3n/2 < 14n/9$. Hence we may assume that $T_u = P_7$ and that $u$ is the central vertex of $T_u$. But then $T = (P_1 \circ P_2) \circ P_2 \in \mathcal{T}$. Suppose secondly that $T_v = P_3$. Then, since $n \leq 9$, $T_u = P_6$. By our assumption following Observation 3, the vertex $u$ is a not a support vertex of $T_u$. But then again $T = (P_1 \circ P_2) \circ P_2 \in \mathcal{T}$.

Suppose finally that $\ell = 2$. Then, $T_v = P_2$ and $T_u \in \{P_2, P_6, P_{10}, P_{14}\}$. Since there is a $\gamma_t(T_u)$-set containing $u$, we have $\gamma_t(T) \leq \gamma_t(T_u) + 1$, implying that $f_t(T) \leq \gamma_t(T_u) + 1 + g_t(T) \leq (|V(T_u)| + 2)/2 + 1 + n - 1 = 3n/2 < 14n/9$. Hence we have established that either $f_t(T) < 14n/9$ or $f_t(T) = 14n/9$ and $T = (P_1 \circ P_2) \circ P_2$. That proves Observation 5. $\Box$

By Observations 4 and 5, we may assume that every edge of $T$ is oriented in exactly one direction. Since $T$ is a tree, it follows that there exist a vertex $v$ with out-degree zero in this oriented tree. Thus for every edge $uv$ in $T$, $T_u \in \mathcal{S}$ and $T_v \notin \mathcal{S}$. If $v$ is a leaf and $u$ the support vertex adjacent with $v$, then $T_v = P_1 \in \mathcal{S}$ in $T - uv$, and so $v$ would have out-degree one in the oriented tree, a contradiction. Hence, $\deg_T(v) \geq 2$.

If every neighbor of $v$ in $T$ has degree at most two we define $I = 0$; otherwise, we define $I = 1$. Applying Lemma 10 to the tree $T$ with $S = \{v\}$, we have $g_t(T) \leq n + 2 - \deg_T(v)$. If $I = 1$, and $u$ is a neighbor of $v$ with $\deg_T(u) \geq 3$, then applying Lemma 10 to the tree $T$ with $S = \{u, v\}$, we have $g_t(T) \leq n + 4 - \deg_T(u) - \deg_T(v) \leq n + 1 - \deg_T(v)$. Hence we have the following observation.

**Observation 6** $g_t(T) \leq n + 2 - \deg_T(v) - I$.

If $v$ is adjacent only to vertices that are isolated in $T - v$ or leaves of a $P_5$ in $T - v$ or the central vertices of a $P_7$ in $T - v$, then we define $J = 1$; otherwise, we define $J = 0$. For a graph $G$, let $\mathrm{oc}(G)$ denote the number of odd components of $G$ and $\mathrm{ec}(G)$ the number of even components of $G$, and let $k_2(G)$ denote the number of $P_2$-components in $G$. Then it follows from Lemmas 1 and 14 that

$$\gamma_t(T) \le \frac{n-1}{2} + \mathrm{ec}(T-v) + \frac{\mathrm{oc}(T-v)}{2} + J,$$

and if $k_2(T-v) \ge 1$, then

$$\gamma_t(T) \le \frac{n-1}{2} + \mathrm{ec}(T-v) + \frac{\mathrm{oc}(T-v)}{2} + 1 - k_2(T-v).$$

Hence, by Observation 6 and since $\deg_T(v) = \mathrm{ec}(T-v) + \mathrm{oc}(T-v)$, we have the following two upper bounds on $f_t(T)$.

**Observation 7** $f_t(T) \le \frac{3n}{2} + \frac{3}{2} - \frac{\mathrm{oc}(T-v)}{2} - I + J.$

**Observation 8** *If $k_2(T-v) \ge 1$, then*

$$f_t(T) \le \frac{3n}{2} + \frac{5}{2} - \frac{\mathrm{oc}(T-v)}{2} - I - k_2(T-v).$$

We proceed further with three observations.

**Observation 9** *If $J = 1$, then $f_t(T) < 14n/9$.*

**Proof.** Suppose $J = 1$. Then $\mathrm{oc}(T-v) = \deg_T(v) \ge 2$. By our assumption following Observation 3 there can be no $P_5$-component of $T-v$. Hence, $v$ is adjacent only to vertices that are isolated in $T-v$ or to the central vertices of a $P_7$ in $T-v$. If $T$ is a star, then the result follows from Lemma 12. Hence we may assume that $v$ is adjacent to the central vertex of a $P_7$ in $T-v$. But then $I = 1$. Thus, by Observation 7, we have that $f_t(T) \le 3n/2 + (3 - \deg_T(v))/2$. If $\deg_T(v) \ge 3$, then $f_t(T) \le 3n/2 < 14n/9$. Hence we may assume that $\deg_T(v) = 2$, and so $f_t(T) \le (3n+1)/2$. If one component of $T-v$ is $P_1$ and the other one is $P_7$ with central vertex $u$, we have a contradicting with the fact that $v$ has out-degree zero in the oriented tree. Hence both components of $T-v$ are $P_7$-components, and so $n = 15$, whence $f_t(T) \le (3n+1)/2 < 14n/9$. □

**Observation 10** *If $I = J = 0$, then $f_t(T) \le 14n/9$ with equality if and only if $T = (P_1 \circ P_2) \circ P_2$.*

**Proof.** Suppose $I = J = 0$. Then every neighbor of $v$ in $T$ has degree at most two. By our assumption following Observation 3 every component of $T - v$ is therefore isomorphic to $P_1, P_2$ and $P_3$ (and so, $\mathrm{ec}(T - v) = k_2(T - v)$). Since $T$ is not a path, $\deg_T(v) \geq 3$. If $T - v$ has no $P_3$-component, then by Lemma 12, $f_t(T) < 14n/9$. Hence we may assume that $T - v$ has a $P_3$-component. If $\mathrm{oc}(T - v) \geq 3$, then by Observation 7, $f_t(T) \leq 3n/2 < 14n/9$. Hence we may assume that $\mathrm{oc}(T - v) \leq 2$. If $k_2(T - v) \geq 2$, then by Observation 8, $f_t(T) \leq 3n/2 < 14n/9$. Hence we may assume that $k_2(T - v) \leq 1$. Thus, since $\deg_T(v) \geq 3$, we have that $\mathrm{oc}(T - v) = 2$ and $k_2(T - v) = 1$. Since $v$ has out-degree zero in the oriented tree, there can be no $P_1$-component in $T - v$. Hence, $T - v$ consists of one $P_2$-component and two $P_3$-components and $v$ is adjacent to a leaf in each of these components. Thus, $T = (P_1 \circ P_2) \circ P_2$. □

**Observation 11** *If $I = 1$ and $J = 0$, then $f_t(T) < 14n/9$.*

**Proof.** Suppose $I = 1$ and $J = 0$. Then, by Observation 7, $f_t(T) \leq 3n/2 + (1 - \mathrm{oc}(T - v))/2$. If $\mathrm{oc}(T - v) \geq 1$, then $f_t(T) \leq 3n/2 < 14n/9$. Hence we may assume that $\mathrm{oc}(T - v) = 0$, and so $f_t(T) \leq (3n + 1)/2$. If $n \leq 9$, then since $v$ by assumption is adjacent to a vertex $u$ of degree at least 3 in $T$, it follows that $T - v = P_2 \cup P_6$. But then if we consider the edge $uv$ we have that $T_v = P_3 \in \mathcal{S}$, contradicting the fact that $v$ has out-degree zero in the oriented tree. Hence, $n > 9$, whence $f_t(T) \leq (3n + 1)/2 < 14n/9$. □

The proof of Theorem 1 now follows from Observations 9, 10 and 11. □

# 5 Proof of Theorem 2

## 5.1 Preliminary Results

**Lemma 15** *If $T$ is a tree of order $n$ that can be obtained from a path $v_1, \ldots, v_{2k+1}$ on $2k + 1$ vertices, where $k \geq 0$, by attaching paths $P_1$ or $P_2$ to vertices in $\{v_1, v_3, \ldots, v_{2k+1}\}$ such that $\deg_T(v_{2i+1}) = 3$ for each $i \in \{0, \ldots, k\}$, then $f_t(T) < 3n/2$.*

**Proof.** We proceed by induction on $k$. If $k = 0$, then $T$ is a star or a subdivided star and the result follows from Lemma 12 and if $k = 1$, then $T$ is one of six small trees (of orders 7, 8, 9, 9, 10, 11) and the result is straightforward to check. This establishes the base cases. Hence we may assume that $k \geq 2$ and that the result of the lemma is true for all trees that can be obtained from a path on $2k' + 1$ vertices where $0 \le k' < k$. Let $T$ be a tree of order $n$ that can be obtained from a path $v_1, \ldots, v_{2k+1}$ on $2k + 1$ vertices by the procedure described in the statement of the lemma.

We now consider the forest $F = T - v_3v_4$. Let $F_1$ and $F_2$ be the components of $F$ containing $v_3$ and $v_4$, respectively. For $i = 1, 2$, let $F_i$ have order $n_i$, and so $n = n_1 + n_2$. Then, $F_1 \ne (P_1 \circ P_2) \circ P_2$ and $F_1$ is a tree with $6 \le n_1 \le 9$, with three leaves, one vertex of degree 3, and with the remaining vertices of degree 2. Thus, by Theorem 1, $f_t(F_1) < 14n_1/9$. Hence, since $6 \le n_1 \le 9$, $f_t(F_1) \le \lfloor (14n_1 - 1)/9 \rfloor \le \lfloor 3n_1/2 \rfloor \le 3n_1/2$. If $k = 2$, then by Lemma 12, $f_t(F_2) < 3n_2/2$. If $k \ge 3$, then we can apply the inductive hypothesis to the tree $F_2$, and so $f_t(F_2) < 3n_2/2$. In both cases, $f_t(F_2) < 3n_2/2$. Hence, $f_t(T) \le f_t(F_1) + f_t(F_2) < 3n/2$. $\square$

**Lemma 16** *For $n \ge 3$, $f_t(C_n) \le 3n/2$ with equality if and only if $n \equiv 0 \,(mod\, 4)$.*

**Proof.** Let $G = C_n$, and let $V_1$ and $V_2$ be a partition of $V(G)$ satisfying $f_t(G) = f_t(G; V_1, V_2)$. Suppose that both $V_1$ and $V_2$ are open packings in $G$. Let $i \in \{1, 2\}$. Since no two vertices of $V_i$ have a common neighbor, every vertex in $G[V_i]$ has degree one and the set of edges $[V_1, V_2]$ therefore induces a matching in $G$. Thus since $G$ is 2-regular, we must have that $|V_1| = |V_2|$, $[V_1, V_2]$ induces a perfect matching in $G$, and that $G[V_i]$ is $K_2$ or the disjoint union of copies of $K_2$. Hence, $n \equiv 0 \,(\mathrm{mod}\, 4)$.

If $n$ is odd, then at least one of the sets $V_1$ and $V_2$ is not an open packing in $G$, and so, by Lemma 6, $\gamma_t(G, V_1) + \gamma_t(G, V_2) \le n - 1$. By Lemma 1, $\gamma_t(C_n) = (n+1)/2$ for $n$ odd. Hence, $f_t(G) \le (3n-1)/2$. Therefore we may assume that $n$ is even.

Suppose $n \equiv 2 \,(\mathrm{mod}\, 4)$. Then, by Lemma 1, $\gamma_t(C_n) = (n + 2)/2$. If $V_1$ or $V_2$ is empty, then $f_t(G) \le 2\gamma_t(C_n) = n + 2 < 3n/2$ since $n \ge 6$. Suppose $|V_1| = 1$. Then, $G[V_2] = P_{n-1}$, and so $\gamma_t(G, V_2) \le \gamma_t(G[V_2], V_2) \le \gamma_t(G[V_2]) = \gamma_t(P_{n-1}) = n/2$, implying that $f_t(G) = \gamma_t(G) + \gamma_t(G, V_1) + \gamma_t(G, V_2) \le (n+2)/2 + 1 + n/2 = n + 2 < 3n/2$. Hence we may assume that $|V_1| \ge 2$ and $|V_2| \ge 2$.

For $i \in \{1, 2\}$, if there are two adjacent vertices with the same label $i$, then $\gamma_t(G, V_{3-i}) \le \gamma_t(P_{n-2}) = (n - 2)/2$. Hence if both sets $V_1$ and $V_2$ contain adjacent vertices, then $f_t(G) = \gamma_t(G) + \gamma_t(G, V_1) + \gamma_t(G, V_2) \le (n + 2)/2 + n - 2 = (3n - 2)/2$. Thus we may assume that at least one of $V_1$ and $V_2$, say $V_1$, is an independent set. This implies that $V_2$ is not an open packing, and so $\gamma_t(G, V_2) \le |V_2| - 1$. If $V_1$ is not an open packing, then $\gamma_t(G, V_1) \le |V_1| - 1$, implying that $f_t(G) \le (n + 2)/2 + |V_1| + |V_2| - 2 = (3n - 2)/2$. Hence we may assume that $V_1$ is both an independent set and an open packing. Thus since the vertices in the set $V_1$ have disjoint neighborhoods in $G$, $N(V_1) \subseteq V_2$ and $|N(V_1)| = 2|V_1|$. For each vertex $v \in V_2 \setminus N(V_1)$, we choose an adjacent vertex and call the resulting set of such vertices $V_2'$. Then, $V_1 \cup V_2'$ totally dominates $V_2$, and so $\gamma_t(G, V_2) \leq |V_1| + |V_2'| \leq |V_1| + |V_2 \setminus N(V_1)| = |V_1| + |V_2| - |N(V_1)| = |V_2| - |V_1|$. Thus since $\gamma_t(G, V_1) = |V_1|$ and $\gamma_t(G) = (n+2)/2$, we have that $f_t(G) \leq (n+2)/2 + |V_2| \leq (n+2)/2 + n - 2 = (3n-2)/2$. Hence if $n \equiv 2 \,(\mathrm{mod}\, 4)$, then $f_t(G) \leq (3n-2)/2 < 3n/2$.

Suppose, finally, that $n \equiv 0 \,(\mathrm{mod}\, 4)$. Then, by Lemma 1, $\gamma_t(C_n) = n/2$. Since there is a good partition of $V(G)$ in this case, $g_t(G) = n$, implying that $f_t(G) = 3n/2$. □

**Lemma 17** *For $n \geq 3$, let $G = C_n$ where $n \equiv 0 \,(mod 4)$, and let $V_1$, $V_2$ be a partition of $V(G)$. Then, $f_t(G; V_1, V_2) \leq 3n/2$ with equality if and only if $V_1$, $V_2$ is a good partition of $V(G)$.*

**Proof.** By Lemma 16, $f_t(G; V_1, V_2) \leq f_t(G) = 3n/2$. If $V_1$, $V_2$ is not a good partition of $V(G)$, then $V_1$ or $V_2$ is not an open packing in $G$, and so, by Lemma 6, $\gamma_t(G, V_1) + \gamma_t(G, V_2) \leq n - 1$. Together with Lemma 1, $\gamma_t(G) = n/2$, we obtain $f_t(G; V_1, V_2) \leq 3n/2 - 1$. Conversely, if $V_1$, $V_2$ is a good partition of $V(G)$, then both $V_1$ and $V_2$ are open packings in $G$, implying by Lemma 6 that $\gamma_t(G, V_1) + \gamma_t(G, V_2) = n$, whence $f_t(G; V_1, V_2) = 3n/2$. □

**Lemma 18** *If $G$ is a graph of order $n$ that can be obtained from a cycle $v_0, v_1, \ldots, v_{2k-1}, v_0$ on $2k$ vertices, where $k \geq 2$, by attaching for each $i \in \{0, 1, \ldots, k-1\}$ a path $P_1$ or $P_2$ to $v_{2i}$, then $f_t(G) < 3n/2$.*

**Proof.** Let $G = (V, E)$. If $k = 2$, then $G$ is one of three graphs (of orders 6, 7 and 8) and the result is straightforward to check. Hence we may assume that $k \geq 3$. Let $i \in \{0, 1, \ldots, k-1\}$ and let $F_i$ and $G_i$ be the components of $G - \{v_{2i-1}v_{2i}, v_{2i+2}v_{2i+3}\}$ containing $v_{2i}$ and $v_{2i-1}$, respectively (where addition is taken modulo $2k$). Then, $F_i$ is a path of order 5, 6 or 7, while $G_i$ is a tree that can be obtained from a path on $2(k-3)+1$ vertices by the procedure described in the statement of the Lemma 15. By Lemma 15, $f_t(G_i) < 3|V(G_i)|/2$.

Let $V_1$, $V_2$ be a partition of $V$ such that $f_t(G; V_1, V_2) = f_t(G)$. For $j = 1, 2$, let $V_{i,j} = V_j \cap V(F_i)$. Suppose that $V_{i,1}$, $V_{i,2}$ is not a good partition of $V(F_i)$. Then, by Lemma 9, $f_t(F_i; V_{i,1}, V_{i,2}) \leq 3|V(F_i)|/2$. Thus, $f_t(G) = f_t(G; V_1, V_2) \leq f_t(F_i; V_{i,1}, V_{i,2}) + f_t(G_i) < 3|V(F_i)|/2 + 3|V(G_i)|/2 = 3n/2$. Hence we may assume that $V_{i,1}$, $V_{i,2}$ is a good partition of $V(F_i)$ for each $i \in \{0, 1, \ldots, k-1\}$, for otherwise the desired result follows.

Suppose that for some $i \in \{0, 1, \ldots, k-1\}$, the small component of $G - v_{2i}$ and the small component of $G - v_{2i+2}$ are isomorphic (either to $P_1$ or $P_2$). For notational convenience, we may assume that the small component of $G - v_0$ and the small component of $G - v_2$ are isomorphic. Let $T_1$ and $T_2$ be the components of $G - \{v_0 v_{2k-1}, v_4 v_5\}$ containing $v_0$ and $v_{2k-1}$, respectively. Then, $T_1$ is a tree with three leaves, with one vertex of degree 3, and with the remaining vertices of degree 2. Since $T_1$ is one of four small trees, and since $V_{i,1}$, $V_{i,2}$ is a good partition of $V(F_i)$ for every $i \in \{0, 1, \ldots, k-1\}$, and in particular for $i = 0, 1$, it is straightforward to check that $f_t(T_1) \le 3|V(T_1)|/2$. If $k = 3$, then $V(T_2) = \{v_5\}$ and since there exists a $\gamma_t(T_1)$-set containing $v_0$, it follows that $f_t(G) \le f_t(T_1) + 1 \le 3(n-1)/2 + 1 < 3n/2$. If $k \ge 4$, then by Lemma 15, $f_t(T_2) < 3|V(T_2)|/2$, implying that $f_t(G) \le f_t(T_1) + f_2(T_2) < 3|V(T_1)|/2 + 3|V(T_2)|/2 = 3n/2$.

Hence we may assume that for every $i \in \{0, 1, \ldots, k-1\}$, the small component of $G - v_{2i}$ and the small component of $G - v_{2i+2}$ are not isomorphic. Thus, $k$ must be even. We may assume that for $i \equiv 0 \pmod 4$, $G - v_i$ has a component isomorphic to $P_2$ (and therefore for $i \equiv 2 \pmod 4$, $G - v_i$ has a component isomorphic to $P_1$). Let $C$ denote the cycle in $G$ (of order $2k$). Let $H$ be the spanning subgraph of $G$ obtained from $G$ by deleting all edges on $C$ incident with vertices $v_i$ where $i \equiv 0 \pmod 4$. Then, $H$ is isomorphic to $k/2$ disjoint copies of $P_3 \cup K_{1,3}$. Hence since $f_t(P_3 \cup K_{1,3}) = 10$, it follows that $f_t(G) \le f_t(H) \le 10|V(H)|/7 = 10n/7 < 3n/2$. $\square$

### 5.2 Notation

Before proceeding with a proof of Theorem 2, we introduce some additional notation. We define a vertex as ***small*** if it has degree $\le 2$, and ***large*** if it has degree more than 2. In a graph $G$, let $L$ denote the set of all its large vertices. Suppose $|L| \ge 1$ and let $C$ be any component of $G - L$; it is a path (possibly, containing only one vertex). If $C$ has only one vertex and that is adjacent to two large vertices, or if $C$ has at least two vertices and the two ends of $C$ are adjacent in $G$ to different large vertices, then we say that $C$ is a ***2-path***. Otherwise, when the ends of $C$ are adjacent to the same large vertex, we say that $C$ is a ***2-handle***.

### 5.3 Proof of Theorem 2

Recall Theorem 2.

**Theorem 2** *If $G$ is a connected graph of order $n$ with $\delta(G) \ge 2$, then $f_t(G) \le 2n/3$ with equality if and only if $G \cong C_n$ where $n \equiv 0 \,(mod\, 4)$.*

**Proof.** We proceed by induction on $\ell = n+m$, where $m$ denotes the size of $G$. Note that $n \geq 3$ and $m \geq 3$, and so $\ell \geq 6$. When $\ell = 6$, the graph $G$ is a 3-cycle and $f_t(G) = 4 < 3n/2$. This establishes the base case. For the inductive hypothesis, let $\ell \geq 7$ and assume for all connected graphs $G'$ of order $n'$ and size $m'$ with $n'+m' < \ell$ and with $\delta(G') \geq 2$ that $f_t(G') \leq 2n'/3$ with equality if and only if $G' \cong C_{n'}$ where $n' \equiv 0 \,(\mathrm{mod}\, 4)$.

So let $G = (V,E)$ be a connected graph of order $n$ and size $m$ with $m+n = \ell$ and with $\delta(G) \geq 2$. Suppose that $G$ contains at least one large vertex. Let $L$ be the set of all large vertices of $G$.

**Observation 12** *If $L$ contains two adjacent vertices, then $f_t(G) < 3n/2$.*

**Proof.** Suppose that two large vertices $u$ and $v$ are adjacent. Let $G' = G - uv$. Then, $G'$ is a graph of order $n' = n$ and size $m' = m-1$ and with $\delta(G') \geq 2$. Applying the inductive hypothesis to every component of $G'$, we have that $f_t(G') \leq 3n'/2 = 3n/2$ with equality if and only if every component of $G'$ is a cycle of order congruent to zero modulo 4. By Lemma 3, $f_t(G) \leq f_t(G') \leq 3n/2$. Thus if $f_t(G') < 3n/2$, then $f_t(G) < 3n/2$. If $f_t(G') = 3n/2$, then every component of $G'$ is a cycle of order congruent to zero modulo 4, and so, by Lemma 1, $\gamma_t(G') = n/2$, whence $\gamma_t(G) \leq n/2$. By Lemma 11, $\gamma_t(G,V_1) + \gamma_t(G,V_2) \leq n-1$ for every partition $V_1$, $V_2$ of $V(G)$. Thus, $f_t(G) \leq 3n/2 - 1$. □

By Observation 12, we may assume that $L$ is an independent set (for otherwise, the desired result follows).

**Observation 13** *If $G$ contains a path on six vertices each internal vertex of which has degree 2 in $G$ and whose end-vertices are not adjacent, then $f_t(G) < 3n/2$.*

**Proof.** Let $u$ and $v$ be the two end-vertices of a path $P$ on six vertices each internal vertex of which has degree 2. Let $G'$ be the graph obtained from $G$ by removing the four internal vertices of this path and adding the edge $uv$. Then, $G'$ is a connected graph of order $n' = n-4$ and size $m' = m-4$ with $\delta(G') \geq 2$. Applying the inductive hypothesis to $G'$, we have that $f_t(G') \leq 3n'/2 = 3n/2 - 6$ with equality if and only if $G'$ is a cycle of order congruent to zero modulo 4. Since the degree of every large vertex of $G$ remains unchanged in $G'$, $\Delta(G') \geq 3$, implying that $f_t(G') < 3n/2 - 6$.

Let $V_1$, $V_2$ be a partition of $V$, and let $P$ be the path $u, u_1, u_2, u_3, u_4, v$. Thus, $G' = (G - \{u_1,u_2,u_3,u_4\}) \cup \{uv\}$. Let $i \in \{1,2\}$ and let $V_i' = V(G') \cap V_i$. Let

$U \subseteq V(G')$ and let $S'$ be a minimum set of vertices in $G'$ that totally dominates $U$ in $G'$, and so $|S'| = \gamma_t(G', U)$. If $\{u, v\} \subseteq S'$, let $S = S' \cup \{u_1, u_4\}$. If $\{u, v\} \cap S' = \emptyset$, let $S = S' \cup \{u_2, u_3\}$. If $u \in S'$ and $v \notin S'$, let $S = S' \cup \{u_3, u_4\}$. If $u \notin S'$ and $v \in S'$, let $S = S' \cup \{u_1, u_2\}$. In all cases, $|S| = |S'| + 2$ and $S$ totally dominates $U \cup V(P)$ in $G$. In particular, if $U = V(G')$, then $S'$ is a $\gamma_t(G')$-set and $S$ is a TDS of $G$, whence $\gamma_t(G) \leq |S| = |S'| + 2 = \gamma_t(G') + 2$. If $U = V_i'$, then $S$ totally dominates $V_i$ in $G$, and so $\gamma_t(G, V_i) \leq |S| = |S'| + 2 = \gamma_t(G', V_i') + 2$. Hence, $f_t(G; V_1, V_2) \leq f_t(G'; V_1', V_2') + 6 \leq f_t(G') + 6 < 3n/2$. Thus for every partition $V_1$, $V_2$ of $V$, $f_t(G; V_1, V_2) < 3n/2$. Therefore, $f_t(G) < 3n/2$. $\Box$

By Observation 13, we may assume that $G$ contains no path on six vertices each internal vertex of which has degree 2 in $G$ and whose end-vertices are not adjacent. Hence since $L$ is an independent set, we have the observation.

**Observation 14** *Every* 2*-path contains at most three vertices, while every* 2*-handle contains at most five vertices.*

**Observation 15** *If $G$ contains a degree-*3 *vertex that is adjacent to the ends of a* 2*-handle, then $f_t(G) < 3n/2$.*

**Proof.** Assume that there is a degree-3 vertex $v$ that is adjacent to the ends of a 2-handle $C$. By Observation 14, $2 \leq |C| \leq 5$. By connectivity there exists a 2-path $P$ with an end adjacent to $v$. Let $u$ be the other large vertex adjacent with an end of $P$. By Observation 14, $1 \leq |P| \leq 3$. Let $G'$ be the spanning subgraph of graph obtained from $G$ by removing the edge joining $u$ with an end of $P$. Let $G_u$ and $G_v$ be the components of $G'$ containing $u$ and $v$, respectively. Let $|V(G_u)| = n_u$ and $|V(G_v)| = n_v$, and so $n = n_u + n_v$. Now, $\delta(G_u) \geq 2$ while $G_v$ is a key $L_{r,s}$ where $r = |C| + 1$ and $s = |P|$. Hence, $3 \leq r \leq 6$ and $1 \leq s \leq 3$. Thus, by Lemma 2, $\gamma_t(G_v) \leq (n_v + 1)/2$. By Lemma 11, $\gamma_t(G_v, V_1) + \gamma_t(G_v, V_2) \leq n_v - 1$ for every partition $V_1$, $V_2$ of $V(G_v)$. Thus, $f_t(G_v) \leq (3n_v - 1)/2$. Applying the inductive hypothesis to the graph $G_u$, $f_t(G_u) \leq 3n_u/2$. Hence, $f_t(G') = f_t(G_u) + f_t(G_v) \leq (3n - 1)/2$. Thus, by Lemma 3, $f_t(G) \leq f_t(G') < 3n/2$. $\Box$

By Observation 15, we may assume that every large vertex in $G$ that is adjacent to the ends of a 2-handle has degree at least 4.

**Observation 16** *If $G$ contains a* 2*-handle of order* 2, 4 *or* 5, *then* $f_t(G) < 3n/2$.

**Proof.** Suppose there is a 2-handle $C$ where $|C| = k$ and $k \in \{2, 4, 5\}$. Say its ends have common neighbor $v \in L$. By assumption, $\deg_G(v) \geq 4$. Let $G' = G - V(C)$. Then, $G'$ is a connected graph of order $n' = n - k$ and size $m' = m - k - 1$ and with $\delta(G') \geq 2$. Applying the inductive hypothesis to $G'$, we have that $f_t(G') \leq 3n'/2 = 3(n-k)/2$ with equality if and only if $G'$ is a cycle of order congruent to zero modulo 4.

Let $V_1$, $V_2$ be a partition of $V$ and for $i \in \{1, 2\}$, let $V_i' = V(G') \cap V_i$. Let $U \subseteq V'(G)$ and let $S'$ be a minimum set of vertices in $G'$ that totally dominates $U$ in $G'$, and so $|S'| = \gamma_t(G', U)$.

Suppose $k = 2$. Then, $S' \cup \{v\}$ totally dominates $U \cup V(C)$ in $G$. It follows that $\gamma_t(G) \leq \gamma_t(G') + 1$, and for $i \in \{1, 2\}$, $\gamma_t(G, V_i) \leq \gamma_t(G', V_i') + 1$. Hence, $f_t(G; V_1, V_2) \leq f_t(G'; V_1', V_2') + 3 \leq f_t(G') + 3 \leq 3n/2$. If $f_t(G') < 3(n-2)/2$, then $f_t(G; V_1, V_2) < 3n/2$. If $f_t(G') = 3(n-2)/2$, then $G'$ is a cycle (congruent to zero modulo 4). But then we can choose a $\gamma_t(G')$-set to contain $v$, implying that $\gamma_t(G) \leq \gamma_t(G')$ and $f_t(G; V_1, V_2) \leq f_t(G'; V_1', V_2') + 2 \leq f_t(G') + 2 \leq 3n/2 - 1$. Thus for every partition $V_1$, $V_2$ of $V$, $f_t(G; V_1, V_2) < 3n/2$. Therefore, $f_t(G) < 3n/2$.

Suppose $k = 4$. Let $C$ be the path $v_1, v_2, v_3, v_4$. Then, $S' \cup \{v_2, v_3\}$ totally dominates $U \cup V(C)$ in $G$. It follows that $f_t(G; V_1, V_2) \leq f_t(G'; V_1', V_2') + 6 \leq f_t(G') + 6 \leq 3n/2$. If $f_t(G') < 3(n-4)/2$, then $f_t(G; V_1, V_2) < 3n/2$. If $f_t(G') = 3(n-4)/2$, then $G'$ is a cycle of order congruent to zero modulo 4, and so, by Lemma 1, $\gamma_t(G') = n'/2 = (n-4)/2$, whence $\gamma_t(G) \leq n/2$. By Lemma 11, $\gamma_t(G, V_1) + \gamma_t(G, V_2) \leq n - 1$, and so $f_t(G; V_1, V_2) \leq 3n/2 - 1$. Thus for every partition $V_1$, $V_2$ of $V$, $f_t(G; V_1, V_2) < 3n/2$. Therefore, $f_t(G) < 3n/2$.

Suppose $k = 5$. Let $C$ be the path $v_1, v_2, v_3, v_4, v_5$. For $i = 1, 2$, let $W_i = V_i \cap V(C)$. If $W_1$, $W_2$ is not a good partition of $V(C)$, then by Lemma 8, $f_t(C; W_1, W_2) \leq 3(k-1)/2 = 7$. Thus,

$$\begin{aligned} f_t(G; V_1, V_2) &\leq f_t(C; W_1, W_2) + f_t(G'; V_1', V_2') \leq 7 + f_t(G') \\ &\leq 7 + 3(n-5)/2 = (3n-1)/2. \end{aligned}$$

On the other hand, suppose that $W_1$, $W_2$ is a good partition of $V(C)$. Thus, renaming the sets $V_1$ and $V_2$ if necessary, we may assume that $W_1 = \{v_1, v_2, v_5\}$ (that is, the labels of $v_1, v_2, v_3, v_4, v_5$ are given by $1, 1, 2, 2, 1$, respectively). But then $\{v, v_1\}$ totally dominates $W_1$ in $G$, $\{v_3, v_4\}$ totally dominates $W_2$ in $G$, and $\{v, v_3, v_4\}$ totally dominates $V(C)$ in $G$. Hence, $f_t(G; V_1, V_2) \leq 7 + f_t(G'; V_1', V_2') \leq 7 + 3(n-5)/2 = (3n-1)/2$. Thus for every partition $V_1$, $V_2$ of $V$, $f_t(G; V_1, V_2) < 3n/2$. Therefore, $f_t(G) < 3n/2$. $\square$

By Observations 14 and 16, we have the observation.

**Observation 17** *Every 2-handle contains three vertices.*

We now construct a spanning subgraph $H$ of $G$ as follows. First from every 2-handle (of order 3) and every 2-path that contains two or three vertices, we delete exactly one edge (both of whose ends necessarily have degree 2). Thus in the resulting graph, there is no 2-handle and every 2-path, if any, has order 1. We then successively delete an edge that joins the single vertex of a 2-path with a large vertex of degree at least 4 in the graph obtained at each stage until no such edge remains. (Thus if a large vertex in the graph constructed at this stage is adjacent with the vertex of a 2-path, then this large vertex has degree 3.) Finally in the resulting graph, we successively delete two of the three edges incident with every large vertex all of whose neighbors are vertices of 2-paths (of order 1) in the resulting graph at each stage until no such large vertex remains. Let $H$ denote the resulting spanning subgraph of $G$.

By construction, $H$ has no 2-handle and every 2-path in $H$, if any, has order 1. Further, every large vertex of $H$ that is adjacent to the vertex of a 2-path has degree 3 and has at least one neighbor (of degree 1 or 2) that is not on any 2-path. (Thus no large vertex is adjacent to the ends of more than two 2-paths.) Each leaf in $H$ is either adjacent to a large vertex of $H$ or is adjacent to a degree-2 vertex that is adjacent to a large vertex of $H$. It follows that every component $H'$ of the spanning subgraph $H$ of $G$ is isomorphic to one of the graphs described in Lemmas 12, 15 or 18: If $H'$ contains only one large vertex, then $H'$ is one of the graphs described in Lemma 12 (stars with possible subdivisions). If the vertices of $H'$ that belong to 2-paths (of order 1) and their neighbors (the large vertices in $H'$) induce a path in $H'$, then $H'$ is one of the graphs described in Lemma 15 (paths with pendants). If the vertices of $H'$ that belong to 2-paths and their neighbors induce a cycle in $H'$, then $H'$ is one of the graphs described in Lemma 18 (cycles with pendants). Hence by Lemma 3, and by Lemmas 12, 15 or 18, it follows that $f_t(G) \leq f_t(H) < 3n/2$.

Hence we have shown that if $G$ contains at least one large vertex, then $f_t(G) < 3n/2$. If $G$ contains no large vertex, then $G$ is a cycle, and the desired result follows from Lemma 16. $\Box$

# References


[1] G. Chartrand and L. Lesniak, *Graphs & Digraphs: Third Edition*, Chapman & Hall, London, 1996.

[2] E. J. Cockayne, R. M. Dawes, and S. T. Hedetniemi, Total domination in graphs. *Networks* **10** (1980), 211–219.

[3] T.W. Haynes, S.T. Hedetniemi, and P.J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, New York, 1998.

[4] T.W. Haynes, S.T. Hedetniemi, and P.J. Slater (eds), *Domination in Graphs: Advanced Topics*, Marcel Dekker, New York, 1998.

[5] B.L. Hartnell and P.D. Vestergaard, Partitions and dominations in a graph. *J. Combin. Math. Combin. Comput.* **46** (2003), 113–128.

[6] M. A. Henning, Graphs with large total domination number. *J. Graph Theory* **35** (2000), 21–45.

[7] M. A. Henning and P. D. Vestergaard, Domination in partitioned graphs with minimum degree two, manuscript (2004).

[8] S.M. Seager, Partition dominations of graphs of minimum degree 2. *Congress. Numer.* **132** (1998), 85–91.

[9] Z. Tuza and P.D. Vestergaard, Domination in partitioned graph. *Discussiones Mathematicae Graph Theory.* **22** (2002), 199–210.

# Total Domination in Partitioned Graphs

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**Abstract**

We present results on total domination in a partitioned graph $G = (V, E)$. Let $\gamma_t(G)$ denote the total dominating number of $G$. For a partition $V_1, V_2, \ldots, V_k$, $k \geq 2$, of $V$, let $\gamma_t(G; V_i)$ be the cardinality of a smallest subset of $V$ such that every vertex of $V_i$ has a neighbour in it and define the following

$$f_t(G; V_1, V_2, \ldots, V_k) = \gamma_t(G) + \gamma_t(G; V_1) + \gamma_t(G; V_2) + \ldots + \gamma_t(G; V_k)$$
$$f_t(G) = \max\{f_t(G; V_1, V_2, \ldots, V_k) \mid V_1, V_2, \ldots, V_k \text{ is a partition of } V\}$$
$$g_t(G) = \max\{\Sigma_{i=1}^{k} \gamma_t(G; V_i) \mid V_1, V_2, \ldots, V_k \text{ is a partition of } V\}$$

We summarize known bounds on $\gamma_t(G)$ and for graphs with all degrees at least $\delta$ we derive the following bounds for $f_t(G)$ and $g_t(G)$.

**(i)** for $\delta \geq 2$ and $k \geq 3$ we prove $f_t(G) \leq 11|V|/7$ and this inequality is sharp

**(ii)** for $\delta \geq 3$ and $k = 2$ we prove that $f_t(G) \leq (5/4 - 1/372)|V|$. That inequality may not be sharp, but we conjecture that $f_t(G) \leq 7|V|/6$ is best possible.

**(iii)** for $\delta \geq 3$ and $k = 3$ we prove $f_t(G) \leq 3|V|/2$ and this inequality is sharp.

**(iv)** for $\delta \geq 3$ and $k = 2$ the inequality $g_t(G) \leq 3|V|/4$ holds and is best possible.

**Keywords:** domination; total domination ; partitioned graph; hypergraph
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# 1 Notation

By $G = (V, E)$ we denote a *graph* $G$ with *vertex* set $V = V(G)$ and *edge* set $E = E(G)$. The *order* of $G$ is $|V(G)| = n$. For $x \in V(G)$ we denote by $N_G(x)$ the set of *neighbours* to $x$ and $N_G[x] = \{x\} \cup N_G(x)$. Indices may be omitted if clear from context. The *degree* of $x$ is $d_G(x) = |N_G(x)|$, the number of neighbours to $x$. We let $\delta(G) = \delta$ denote the *minimum degree* in $G$ and $\Delta(G) = \Delta$ the *maximum* degree. A *hypergraph* $H = (V, E)$ has *vertex set* $V = V(H)$ and its set of *hyperedges*, or *edges* for short, is $E = E(H)$. Each hyperedge $e$ is a subset of $V$, $e \subseteq V(H)$. A vertex $v$ is *incident* with an edge $e$ if $v \in e$, the *degree* of $v$ is the number of hyperedges in $H$ containing $v$. We let $\delta(H) = \delta$ denote the *minimum degree* in $H$ and $\Delta(H) = \Delta$ the *maximum degree. H* is *$r$-regular* if each vertex has degree $r$, i.e. $d_H(x) = r$, or equivalently, $x$ is contained in precisely $r$ edges. $H$ is *$k$-uniform* if each hyperedge contains exactly $k$ vertices. Two edges $e_1$ and $e_2$ are said to be *overlapping* if $|V(e_1) \cap V(e_2)| \geq 2$. Let $Y \subseteq V(H)$ then $E(Y)$ denotes all hyperedges, $e$, contained in $Y$ (i.e. $V(e) \subseteq Y$).

For a hypergraph $H$ a *hitting set* or a *transversal* $\mathcal{T}$ is a set of vertices $\mathcal{T} \subseteq V(H)$ such that $e \cap \mathcal{T} \neq \emptyset$ for each hyperedge $e$ in $E(H)$, i.e. each edge $e$ contains at least one vertex from $\mathcal{T}$. $\mathcal{T}(H)$ denotes the minimum cardinality of a transversal for the hypergraph $H$. For sets $S, T \subseteq V$, in a graph $G$ the set $S$ *totally dominates* $T$ if every vertex in $T$ is adjacent to some vertex of $S$. The minimum number of vertices needed to totally dominate $V$ is the *total domination number* $\gamma_t(G)$. For a subset $S$ of $V$ we let $\gamma_t(G; S)$ denote the smallest number of vertices in $G$ which totally dominates $S$. A *partition* $V = (V_1, V_2, \ldots, V_k)$ of $V(G)$ into $k$ disjoint sets, $k \geq 2$, has $V = \bigcup_{i=1}^{k} V_i$, $V_i \cap V_j = \emptyset$, $1 \leq i < j \leq k$. For a partition $(V_1, V_2, \ldots, V_k)$ of $V$, we define the following.

$$f_t(G; V_1, V_2, \ldots, V_k) = \gamma_t(G) + \gamma_t(G; V_1) + \gamma_t(G; V_2) + \ldots + \gamma_t(G; V_k)$$
$$g_t(G; V_1, V_2, \ldots, V_k) = \gamma_t(G; V_1) + \gamma_t(G; V_2) + \ldots + \gamma_t(G; V_k)$$

We furthermore define $f_t(G)$ and $g_t(G)$ as follows.

$$f_t(G) = \max\{f_t(G; V_1, V_2, \ldots, V_k) \mid V_1, V_2, \ldots, V_k \text{ is a partition of } V\}$$
$$g_t(G) = \max\{g_t(G; V_1, V_2, \ldots, V_k) \mid V_1, V_2, \ldots, V_k \text{ is a partition of } V\}$$

For further notation we refer to Chartrand and Lesniak [1].

# 2 Introduction

The theory of domination is outlined in two books by Haynes, Hedetniemi and Slater [5, 6]. A combination of domination and partitions is treated by Hartnell and Vestergaard [7], Seager [14], Tuza and Vestergaard [17], Henning and Vestergaard [11]. There has been an upsurge in the study of total domination. New results on total domination are given by Henning, Kang, Shan, Thomassé and Yeo in [10, 12, 15, 18]. In [9] Henning surveys recent results on total domination. Here we shall study total domination in partitioned graphs.

# 3 Bounds on $\gamma_t$

We summarize in Theorem 1 results found by Henning, Thomassé and Yeo.

**Theorem 1** *Let $G$ be a connected graph with $n$ vertices and minimum degree $\delta(G) = \delta$. Then*

$\delta \geq 2$ *implies* $\gamma_t(G) \leq 4n/7$ *for* $G \notin \{C_3, C_5, C_6, C_{10}\}$ *([8, Corollary 6]).*

$\delta \geq 3$ *implies* $\gamma_t(G) \leq n/2$. *([15]).*

$\delta \geq 4$ *implies* $\gamma_t(G) \leq 3n/7$ *([15]) and there exists some* $\epsilon > 0$ *such that* $\gamma_t(G) \leq (3/7 - \epsilon)n$ *for* $G \neq G_{14}$*, where* $G_{14}$ *is an incidence bipartite graph of order 14 derived from the Fano plane ([19]).*

It is a conjecture that $\delta \geq 5$ implies $\gamma_t(G) \leq 4n/11$.

Theorem 2 and Theorem 3 below, give conditions for equality in Theorem 1.

**Theorem 2** ([9, Theorem 29]) *Let $G$ be a connected graph of order $n > 14$ with $\delta \geq 2$. Then $\gamma_t(G) = 4n/7$ if and only if $G$ can be obtained from a connected graph $F$ of order at least three by adding $|V(F)|$ disjoint copies of $C_6$, one corresponding to each $v \in V(F)$, such that either $v$ is joined by a new edge to a vertex in its corresponding $C_6$ or by two new edges to two vertices at distance two apart in its corresponding $C_6$.*

The family $\mathcal{G} \cup \mathcal{H}$ is constructed in [3] as follows:
Take two copies $a_1b_1a_2b_2 \ldots a_kb_k$ and $c_1d_1c_2d_2 \ldots c_kd_k$, of the path $P_{2k}, k \geq 2$, and add edges $a_id_i$, $b_ic_i$ for $i = 1, 2, \ldots, k$. From this the graph of order $4k$ belonging to the infinite family $\mathcal{G}$ is obtained by adding $a_1c_1$ and $b_kd_k$, while

the graph of order $4k$ in $\mathcal{H}$ is obtained by adding $a_1b_k$ and $c_1d_k$, The generalized Petersen graph $GP_{16}$ is obtained from two circuits $u_1u_2u_3\dots u_7u_8$ and $v_1v_2v_3\dots v_7v_8$ by addition of edges $u_1v_1, u_2v_4, u_3v_7$, $u_4v_2, u_5v_5$, $u_6v_8, u_7v_3, u_8v_6$.

**Theorem 3** ([12, Theorem 5]) *Let $G$ be a connected graph with $\delta(G) \geq 3$. Then $\gamma_t(G) = n/2$ if and only if $G \in \mathcal{G} \cup \mathcal{H}$ or $G = GP_{16}$.*

# 4 $f_t$ for $k$-partitioned graphs with $\delta \geq 2$

We have that $f_t$ increases with the number of partition classes. Therefore we get a weaker inequality if we partition $V$ into more than two classes. That is demonstrated in Theorem 4 below.

**Theorem 4** *Let $G$ be a connected graph of order $n$ with $\delta(G) \geq 2$ and $G \notin \{C_3, C_5, C_6, C_{10}\}$. Let $V(G) = V$ be partitioned into $k$ classes, $k \geq 2$. Then $f_t(G) \leq 11n/7$.*

**If** $k = 2$ *then $f_t(G) \leq 3n/2$. Equality holds if and only if $G$ is a circuit of length zero modulo four, $G = C_{4t}, t \geq 1$.*

**If** $k = 3$ *then $f_t(G) \leq 11n/7$. For $n > 14$ equality holds if and only if $G$ can be obtained from a circuit or a path of order at least three by joining each of its vertices by one edge to disjoint copies of $C_6$.*

**If** $k \geq 4$ *then $f_t(G) \leq 11n/7$ and for $n > 14$ equality holds if and only if $\Delta(G) \leq k$ and $G$ can be obtained from a connected graph $F$ having order at least three and $g_t(F) = |V(F)|$ by adding disjoint copies of $C_6$, one corresponding to each $v \in V(F)$, such that either $v$ is joined by a new edge to one vertex in its corresponding $C_6$ or by two new edges to two vertices at distance two apart in its corresponding $C_6$.*

**Proof.** By Theorem 1 we have $\gamma_t(G) \leq 4n/7$ and assigning to each vertex its own class dominator we have $g_t(G) \leq n$. Therefore $f_t(G) = \gamma_t(G) + g_t(G) \leq 11n/7$. The result for $k = 2$ is proven by Frendrup, Henning and Vestergaard in [4, Theorem 2]. For $k \geq 3$ the equality $f_t(G) = 11n/7$ implies $\gamma_t(G) = 4n/7$ and $g_t(G) = n$ and therefore $G$ has the structure described in Theorem 2. Since $g_t(G) = n$ each subgraph $H$ of $G$ must satisfy $g_t(H) = |V(H)|$ and further $\Delta(G) \leq k$. Let $H_1$ be the graph obtained from a circuit $C_6 : v_1v_2\dots v_6$ by adding a new vertex $x$ and the edge $xv_1$ and let $H_2 := H_1 + xv_3$. Observe for $k = 3$ that $g_t(H_1) = |V(H_1)|$ (obtainable from partitioning $x, v_1, v_2 \dots, v_6$ into classes indexed 1122133 or 1221133) while $g_t(H_2) < |V(H_2)|$. For $k \geq 4$ we can easily show that $g_t(H_i) = |V(H_i)|$, $i = 1, 2$. This proves for $k \geq 3$ that $f_t(G) = 11n/7$ implies $G$ has the structure described in this theorem. Conversely, assume first that $k = 3$ and that $G$ is obtainable as a disjoint union of $H_1$'s with edges added between the vertices named $x$, so they span $F$, where $F$ is a path or circuit. We must exhibit a partition of $V(G)$ proving that $f_t(G) = 11n/7$, i.e. that $g_t(G) = |V(G)|$. It is easy to find a partition $V_1', V_2', V_3'$ of $V(F)$ such that $g_t(F) = |V(F)|$. If $k = 3$ we can extend this partition to all the $H_1$'s such that the following holds, which proves that $g_t(G; V_1', V_2', V_3') = n$.

- $N(x) = N_F(x) \cup \{v_1\}$ contains at most one vertex from each $V_1', V_2', V_3'$ (just put $v_1$ in the partition set which doesn't contain any of the two vertices in $N_F(x)$).
- $N(v_1) = \{x, v_2, v_6\}$ contains one vertex from each $V_1', V_2', V_3'$ (just put $v_2$ and $v_6$ in the partition sets such that this holds).
- $N(v_3), N(v_5) \subset \{v_2, v_4, v_6\}$, which contains one vertex from each $V_1', V_2', V_3'$ (just put $v_4$ in the same set as $x$).
- $N(v_2), N(v_4), N(v_6) \subset \{v_1, v_3, v_5\}$, which contains one vertex from each $V_1', V_2', V_3'$ (just put $v_3$ and $v_5$ in the partition sets such that this holds).

Assume next that $k \geq 4$. Then a vertex $x \in F$ may belong to a unit $H_1$ or $H_2$. Again there is a partition $V_1', V_2', \ldots, V_k'$ of $V(F)$ such that $g_t(F) = |V(F)|$ and similarly to above we can extend this partition to all of $G$, such that the neighbourhood of every vertex in $G$ contains at most one vertex from any partition set. The details are left to the reader. This proves that $g_t(G) = n$. □

## 5 $g_t$ for two-partitioned graphs with $\delta \geq 3$

Chvátal and McDiarmid [2] and Tuza [16] independently established the following result about transversals in hypergraphs (see also Thomassé and Yeo [15] for a short proof of this result).

**Theorem 5** ([2, 16, 15]) *If $H$ is a hypergraph with all edges of size at least three, then $\mathcal{T}(H) \leq (|V(H)| + |E(H)|)/4$.*

**Theorem 6** *Let $G$ be a graph of order $n$ with $\delta \geq 3$ and let $G$ be partitioned into two classes. Then $g_t(G) \leq 3n/4$.*

**Proof.** From the two-partitioned graph $G$, we define for $i = 1, 2$, $H_i$ to be the hypergraph on $n$ vertices and $m_i$ edges where $V(H_i) = V(G)$ and the hyperedges of $H_i$ are the sets of neighbourhoods of class $i$ vertices. In other words, $e \in E(H_i)$ precisely if, for some vertex $v$ in $V_i$, $e = N_G(v)$. Each edge in $H_i$ has at least three vertices because $\delta(G) \geq 3$. In $G$ we see that a set $\mathcal{T}_i$ of vertices totally dominates $V_i$ if and only if $\mathcal{T}_i$ is a transversal of $H_i$. Applying Theorem 5 to $H_1$ and $H_2$ separately we obtain transversals $\mathcal{T}_i$ of $H_i$, $i = 1, 2$, satisfying

$$|\mathcal{T}_1| \leq \tfrac{m_1+n}{4} \qquad\qquad |\mathcal{T}_2| \leq \tfrac{m_2+n}{4}.$$

Since $m_1 + m_2 = n$ we obtain $|\mathcal{T}_1| + |\mathcal{T}_2| \leq \frac{m_1+n}{4} + \frac{m_2+n}{4} = \frac{3n}{4}$. This proves Theorem 6. □

An example of graphs with equality $g_t(G) = 3n/4$ is given in the next section.

## 6 An infinite family of graphs extremal for Theorem 6

We have the following theorem.

**Theorem 7** *For each integer $r \geq 1$ there exists a connected bipartite graph $G_r$ of order $n = 16r$ with $\delta(G_r) = 3$ which for $V(G_r)$ partitioned into two classes has $g_t(G_r) = 3|V(G_r)|/4$ and $f_t(G_r) \geq 9|V(G_r)|/8$.*

**Proof.** We define the graph $G_r$ as follows. Define the vertex set of $G_r$ to be $V(G_r) = W_r \cup A_r \cup B_r$, where

$$\begin{aligned} W_r &= \{w_0, w_1, w_2, \ldots, w_{8r-1}\} \\ A_r &= \{a_0, a_1, a_2, \ldots, a_{4r-1}\} \\ B_r &= \{b_0, b_1, b_2, \ldots, b_{4r-1}\} \end{aligned}$$

We define the edge set of $G_r$ such that the following holds, for all $i \in \{0, 1, 2, \ldots, r-1\}$ (where $b_{-1} = b_{4r-1}$ by definition):

$$\begin{array}{rclcrcl} N(w_{8i}) &=& \{a_{4i}, a_{4i+1}, b_{4i}\} & & N(w_{8i+1}) &=& \{a_{4i}, a_{4i+1}, b_{4i}\} \\ N(w_{8i+2}) &=& \{a_{4i}, a_{4i+2}, b_{4i}\} & & N(w_{8i+3}) &=& \{a_{4i+1}, a_{4i+2}, b_{4i-1}\} \\ N(w_{8i+4}) &=& \{a_{4i+2}, b_{4i+1}, b_{4i+2}\} & & N(w_{8i+5}) &=& \{a_{4i+3}, b_{4i+1}, b_{4i+2}\} \\ N(w_{8i+6}) &=& \{a_{4i+3}, b_{4i+1}, b_{4i+3}\} & & N(w_{8i+7}) &=& \{a_{4i+3}, b_{4i+2}, b_{4i+3}\} \end{array}$$

We now assume $r \geq 1$ is fixed, and therefore omit the subscripts of the above sets and graph. Define $V_1$ and $V_2$ as follows.

$$\begin{aligned} V_1 &= A \cup \cup_{i=0}^{r-1} \{w_{8i+1}, w_{8i+2}, w_{8i+3}, w_{8i+5}\} \\ V_2 &= B \cup \cup_{i=0}^{r-1} \{w_{8i}, w_{8i+4}, w_{8i+6}, w_{8i+7}\} \end{aligned}$$

We will now show that if $S_i$ is a set such that every vertex in $V_i$ has a neighbour in $S_i$, then $|S_i| \geq 3|V(G)|/8$, for $i = 1, 2$. This would imply that $f_t(G) \geq 9|V(G)|/8$ and $g_t(G) \geq 6|V(G)|/8$ when $k = 2$ (as clearly the above would also imply that $\gamma_t(G) \geq 3|V(G)|/8$). From Theorem 6 follows that $g_t(G) = 3|V(G)|/4$.

Let $S_1$ be a set that totally dominates $V_1$ (i.e. every vertex in $V_1$ has a neighbour in $S_1$). As $w_{8i+5}$ has a neighbour in $S_1$ we note $|S_1 \cap \{a_{4i+3}, b_{4i+1}, b_{4i+2}\}| \geq 1$, for all $i = 0, 1, 2, \dots, r-1$. As $w_{8i+1}$, $w_{8i+2}$ and $w_{8i+3}$ all have a neighbour in $S_1$ we note that $|S_1 \cap \{a_{4i}, a_{4i+1}, a_{4i+2}, b_{4i}, b_{4i-1}\}| \geq 2$, for all $i = 0, 1, 2, \dots, r-1$ (recall that $b_{-1} = b_{4r-1}$). As the above sets are all disjoint we note that $|S_1 \cap (A \cup B)| \geq 3|A \cup B|/8$.

As $a_{4i+3}$ has a neighbour in $S_1$ we note that $|S_1 \cap \{w_{8i+5}, w_{8i+6}, w_{8i+7}\}| \geq 1$, for all $i = 0, 1, 2, \dots, r-1$. As $a_{4i}$, $a_{4i+1}$ and $a_{4i+2}$ all have a neighbour in $S_1$ we note that $|S_1 \cap \{w_{8i}, w_{8i+1}, w_{8i+2}, w_{8i+3}, w_{8i+4}\}| \geq 2$, for all $i = 0, 1, 2, \dots, r-1$. As the above sets are all disjoint we note that $|S_1 \cap W| \geq 3|W|/8$. This implies the desired result for $S_1$.

The fact that if $S_2$ totally dominates $V_2$, then $|S_2| \geq 3|V(G)|/8$ is proved analogously to above. We now just need to show that $G$ is connected. Let $Q_i = \{a_{4i}, a_{4i+1}, a_{4i+2}, a_{4i+3}, b_{4i}, b_{4i+1}, b_{4i+2}, b_{4i+3}\}$ and $P_i = \{w_{8i}, w_{8i+1}, \dots, w_{8i+7}\}$ for all $i = 0, 1, 2, \dots, r-1$. Note that $G[P_i \cup Q_i]$ is connected. As the edges $w_{8i+3}b_{4i-1}$, for all $i = 0, 1, 2, \dots, r-1$ connects $P_i$ with $Q_{i-1}$ ($Q_{-1} = Q_{r-1}$) we are done. □

# 7 $f_t(G)$ for two-partitioned graphs with $\delta \geq 3$

Let $G$ be a graph of order $n$ with $\delta(G) \geq 3$ and let its vertices be partitioned into two sets.

From Theorems 1 and 6 it follows immediately that $f_t(G) = \gamma_t(G) + g_t(G) \leq n/2 + 3n/4 = 5n/4$ when $\delta(G) \geq 3$. We shall in Theorem 8 below prove a slightly stronger result and later pose an even stronger conjecture.

The following result is known (see for example [13]).

**Lemma 1** *([13]) If $G$ is a 3-regular graph, then there exists a matching $M$ in $G$, such that $|M| \geq \frac{7}{16}|V(G)|$.*

**Lemma 2** *Let $H$ be a 2-regular 3-uniform hypergraph with no two edges overlapping. Then $\mathcal{T}(H) \leq \frac{|V(H)+|E(H)|}{4} - \frac{|V(H)|}{24}$.*

**Proof.** Let $H$ be a 2-regular 3-uniform hypergraph with no overlapping edges. Define the graph $G_H$ as follows:
$V(G_H) = E(H)$ and $E(G_H) = \{e_1e_2 : |V(e_1) \cap V(e_2)| = 1\}$. As there are no overlapping edges and $H$ is 2-regular and 3-uniform, we note that $G_H$ is a 3-regular graph. By Lemma 1, there exists a matching $M$ in $G_H$, such that $|M| \geq \frac{7}{16}|V(G_H)|$.

If $e_1e_2 \in M$, then by the definition of $G_H$ we note that $V(e_1) \cap V(e_2) = \{x_{e_1e_2}\}$ for some $x_{e_1e_2} \in V(H)$. Let $X = \{x_f \mid f \in M\}$ and note that $2|M|$ edges in $H$ contain a vertex from $X$ (as $M$ was a matching). Let $X'$ be a set of vertices of order $|E(H)| - 2|M|$ containing a vertex from every edge in $H$, which does not contain a vertex from $X$. Note that $X \cup X'$ is a transversal of $H$ of order $|M| + (|E(H)| - 2|M|)$. By the above bound on $|M|$ we get the following, as $3|E(H)| = \sum_{x \in V(H)} d(x) = 2|V(H)|$.

$$\begin{array}{rcl}
\mathcal{T}(H) & \leq & |E(H)| - |M| \leq |E(H)| - \frac{7}{16}|E(H)| \\
 & = & \frac{|E(H)|}{4} + \frac{5|E(H)|}{16} = \frac{|E(H)|}{4} + \frac{5}{16} \times \frac{2|V(H)|}{3} \\
 & = & \frac{|V(H)|+|E(H)|}{4} - \frac{|V(H)|}{24}
\end{array}$$

□

**Lemma 3** *Let $H$ be a 3-uniform hypergraph, where multiple edges are allowed. For each edge and vertex in $H$ we assign a non-empty subset of $\{0, 1, 2\}$. Let this subset be denoted by $L(q)$ for all $q \in V(H) \cup E(H)$. Let $H_i$ be the 3-uniform hypergraph containing vertex-set $V_i = \{v : i \in L(v)$ and $v \in V(H)\}$ and edge-set $E_i = \{e : i \in L(v)$ and $e \in E(H)\}$, for $i = 0, 1, 2$. Let $Y \subseteq V(H)$ be arbitrary and assume that the following holds.*

**(a):** $\Delta(H_1), \Delta(H_2) \leq 2$

**(b):** $\Delta(H - E(Y)) \leq 4$.

**(c):** *There are no overlapping edges in $H_i$, $i \in \{1, 2\}$.*

**(d):** *If $e \in E(H) - E(Y)$, then $0 \in L(e)$ and $|L(e)| \geq 2$.*

*This implies that the following holds.*

$$\sum_{i=0}^{2} \mathcal{T}(H_i) \leq \left( \sum_{i=0}^{2} \frac{|V_i| + |E_i|}{4} \right) - \frac{|V(H_0) \cap V(H_1) \cap V(H_2) \setminus N_H[Y]|}{372}$$

**Remark.** We assume here in Lemma 3 that the assignment of a set $L(q)$ to each $q$ is done such that $H_0, H_1, H_2$ really are hypergraphs, i.e., such that each hyperedge in $E_i$ consists of vertices from $V_i$, $i = 0, 1, 2$. This requirement will be satisfied in the proof of Theorem 8 where the lemma is applied.

**Proof.** Assume that the lemma is false, and that $H$ is a counterexample with minimum $|E_0| + |E_1| + |E_2|$. Clearly $|E_0| + |E_1| + |E_2| > 0$, as otherwise $\sum_{i=0}^{2} \mathcal{T}(H_i) = 0$. For simplicity we will use the following notation:

$$\begin{aligned} T^* &= \textstyle\sum_{i=0}^{2} \mathcal{T}(H_i) \\ S^* &= \textstyle\sum_{i=0}^{2} \frac{|V_i|+|E_i|}{4} \\ V^* &= V(H_0) \cap V(H_1) \cap V(H_2) \end{aligned}$$

We recall that $H$ was assumed to be a "minimal" counterexample to $T^* \leq S^* - (|V^* \setminus N_H[Y]|)/372$. We will now prove a few claims, which end in a contradiction, thereby proving the lemma. For $H$ the left hand side of the inequality, $\ell$, and the right hand side of the inequality, $r$, in Lemma 3 satisfies $\ell > r$. We shall construct smaller $H'$ which also satisfies (a)-(d) and which therefore has $\ell' \leq r'$ by the minimality of $H$. $H'$ is to be constructed such that there exist $\alpha \leq \beta$ for which $\ell - \alpha \leq \ell'$ and $r' \leq r - \beta$. Those inequalities combine to give the desired contradiction $\ell \leq r$.

*Claim A: If we add a vertex to $Y$, then $N[Y]$ does not increase by more than 9 vertices.*

*Proof of Claim A:* This follows from the fact that $H$ is 3-uniform and $\Delta(H - E(Y)) \leq 4$, by (b) in the statement of the lemma.

*Claim B: There is no $e = \{v_1, v_2, x\} \in E_i$, such that $d_{H_i}(v_1) = d_{H_i}(v_2) = 1$ and $d_{H_i}(x) = 2$, for $i = 0, 1, 2$.*

*Proof of Claim B:* Assume that there is such an edge $e = \{v_1, v_2, x\} \in E_i$. Let $e' = \{w_1, w_2, x\}$ be the other edge in $H_i$ containing $x$. Now delete $v_1$, $v_2$, $x$, $e$ and $e'$ from $H_i$ and add $\{v_1, v_2, x, w_1, w_2\}$ to $Y$. Note that (a)-(d) still hold and that $T^*$ decreases by 1 as we simply add $x$ to any transversal in the new $H_i$ in order to get a transversal in the old $H_i$. By Claim A the set $N[Y]$ does not increase by more than 45 vertices. As $V^*$ does not decrease by more than 3 vertices and $S^*$ decreases by $5/4$, we are done by the "minimality" of $H$ (as $\alpha = 1 \le 5/4 - 48/372 = \beta$ in the argument above Claim A).

*Claim C: There is no $e = \{x, v_1, v_2\} \in E_i$, such that $d_{H_i}(v_1) = d_{H_i}(v_2) = 2$ and $d_{H_i}(x) = 1$, for $i = 1, 2$.*

*Proof of Claim C:* Assume that there is such an edge $e = \{x, v_1, v_2\} \in E_i$. Let $e_1 = \{w_1, w_2, v_1\}$ be the other edge in $H_i$ containing $v_1$ and let $e_2 = \{u_1, u_2, v_2\}$ be the other edge in $H_i$ containing $v_2$. As there are no overlapping edges in $H_i$ (by (c) in the statement of the lemma) we note that $e_1 \neq e_2$ and $|\{w_1, w_2, u_1, u_2\}| \ge 3$. Let $S$ be any subset of $\{w_1, w_2, u_1, u_2\}$ such that $|S| = 3$. We now separately consider the cases when addition of $S$ as a new hyperedge to $H_i$ causes overlapping edges in $H_i$, and when it doesn't.

Assume that adding $S$ to $E_i$ does not cause overlapping edges in $H_i - e_1 - e_2$. Now delete $x$, $v_1$, $v_2$, $e$, $e_1$ and $e_2$ from $H_i$ and add the edge $S$ to $H_i$ (and $H$). Furthermore add $\{x, v_1, v_2, w_1, w_2, u_1, u_2\}$ to $Y$. Note that (a)-(d) still hold. If $T'$ is a transversal in the new $H_i$ then due to the edge $S$ we either have $\{u_1, u_2\} \cap T' \neq \emptyset$, in which case $T' \cup \{v_1\}$ is a transversal in the old $H_i$ or $\{w_1, w_2\} \cap T' \neq \emptyset$, in which case $T' \cup \{v_2\}$ is a transversal in the old $H_i$. Therefore $T^*$ decreases by at most one. By Claim A we have that $N[Y]$ does not increase by more than 63 vertices. As $V^*$ does not decrease by more than 3 and $S^*$ decreases by $5/4$, we are done by the "minimality" of $H$ (as $1 \le 5/4 - 66/372$).

So now assume that the above addition of $S$ would cause overlapping edges in $H_i - e_1 - e_2$. This can only happen if there is an edge $e' \in E_i$ such that $|S \cap V(e')| \ge 2$. Note that by (a) the degree in $H_i$ is two for all vertices in $S \cap V(e')$ (they only lie in $S$ and $e'$). Now delete the vertices $\{x, v_1, v_2\} \cup (S \cap V(e'))$ from $H_i$ and delete the edges $e$, $e_1$, $e_2$ and $e'$ from $H_i$ (do not add the edge $S$ to $H_i$). Furthermore add $\{x, v_1, v_2, w_1, w_2, u_1, u_2\} \cup (V(e') - S)$ to $Y$. Note that (a)-(d) still hold. By a similar argument to above we note that $T^*$ decreases by at most two. By Claim A we see that $N[Y]$ does not increase by more than 72 vertices. As $V^*$ does not decrease by more than 6 and $S^*$ decreases by at least $9/4$, we are done by the "minimality" of $H$ (as $2 \le 9/4 - 78/372$).

*Claim D: There is no $e = \{x, v_1, v_2\} \in E_0$, such that $d_{H_0}(v_1) = d_{H_0}(v_2) = 2$ and $d_{H_0}(x) = 1$ and $|N_{H_0}[V(e)]| \ge 6$.*

*Proof of Claim D:* Assume that there is such an edge $e = \{x, v_1, v_2\} \in E_0$. Let $e_1 = \{w_1, w_2, v_1\}$ be the other edge in $H_0$ containing $v_1$ and let $e_2 = \{u_1, u_2, v_2\}$ be the other edge in $H_0$ containing $v_2$. If $e_1 = e_2$, then $|N_{H_0}[V(e)]| \le 4$, a contradiction. So assume that $e_1 \neq e_2$. As $|N_{H_0}[V(e)]| \ge 6$ we note that

$|\{w_1, w_2, u_1, u_2\}| \geq 3$. We are now done analogously to Claim C.

*Claim E:* $\Delta(H_1), \Delta(H_2) \leq 1$.

*Proof of Claim E:* Assume that $\Delta(H_1) \geq 2$. By (a) we have $\Delta(H_1) = 2$. By Claim B and Claim C we note that there is a 2-regular component, $R$, in $H_1$. There are no overlapping edges in $R$ by (c). By Lemma 2 there is a transversal $T_R$ in $R$ of order at most $(|V(R)| + |E(R)|)/4 - |V(R)|/24$. So delete all edges and vertices in $R$ and add all vertices in $R$ to $Y$. By Claim A we have that $N[Y]$ increases by at most $9|V(R)|$ vertices. We now have a contradiction to the "minimality" of $H$, as $|V(R)|/24 \geq 9|V(R)|/372$. Analogously we can show that $\Delta(H_2) \leq 1$.

*Claim F: Assume $e_1, e_2 \in E(H_0)$ overlap and $e_i = (x_1, x_2, u_i)$ for $i = 1, 2$, where $u_1 \neq u_2$. If $d_{H_0}(x_1) = d_{H_0}(x_2) = 2$, then there is an edge $e' \in E(H_0)$ such that $\{u_1, u_2\} \subseteq V(e')$.*

*Proof of Claim F:* Let $e_1$ and $e_2$ be defined as in the Claim, and assume that there is no edge $e' \in E(H_0)$ such that $\{u_1, u_2\} \subseteq V(e')$. Delete $e_1$, $e_2$, $x_1$, $x_2$ and $u_1$ from $H_0$. For every edge, $e''$, in $H_0$ that contains $u_1$, delete $e''$ and add the edge $(e'' - \{u_1\}) \cup \{u_2\}$ instead. Furthermore add $\{x_1, x_2, u_1, u_2\}$ and $V(e'')$ from all transformed edges, to $Y$. As there is at most 4 edges containing $u_1$ in $H_0 - E(Y)$ we note that $Y$ increases by at most 10 (the neighbours of $u_1$ in $H_0 - E(Y)$ and $\{u_1, u_2\}$). Therefore $V^* - N[Y]$ decreases by at most $3 + 90$, by Claim A. We also note that $S^*$ decreases by $5/4$.

We now show that $T^*$ decreases by at most one. Let $T'$ be a transversal in the new $H_0$. If $u_2 \in T'$ then $T' \cup \{u_1\}$ is a transversal in the old $H_0$. If $u_2 \notin T'$ then $T' \cup \{x_1\}$ is a transversal in the old $H_0$. As (a)-(d) still holds after the above operations, we have a contradiction to the "minimality" of $H$, as $1 \leq 5/4 - 93/372$.

*Definition G: Let $x \in V^* - N[Y]$ be arbitrary. The vertex $x$ exists since otherwise we would be done by Theorem 5.*

*Claim H: $d_{H_1}(u) = d_{H_2}(u) = 1$ for all $u \in N_{H_0}[x]$, where $x$ is defined in Definition G.*

*Proof of Claim H:* Assume that $u \in N_{H_0}[x]$ has $d_{H_2}(u) = 0$ or $u \notin V(H_2)$, which are the only possibilities for $u$, if $d_{H_2}(u) \neq 1$ (by Claim E). If $u \in V(H_2)$ and $d_{H_2}(u) = 0$, then delete $u$ from $V(H_2)$. We are now done as $T^*$ is unchanged, $S^*$ decreases by $1/4$ and $V^* - N[Y]$ does not decrease by more than one. So we may assume that $u \notin V(H_2)$. Since $x \in V^*$ we note that $x \in V(H_1)$ and

$x \in V(H_2)$, which by the above argument implies that $d_{H_1}(x) = d_{H_2}(x) = 1$ and $u \neq x$. Let $e_1 = \{x, u, q\}$ be the edge in $H_1$ (and $H_0$) containing $u$ and $x$. Let $e_2$ be the edge in $H_2$ (and $H_0$) that contains $x$. Note that $d_{H_0}(x) = 2$ and $d_{H_0}(u) = 1$. If $d_{H_0}(q) = 1$ then we are done by Claim B. So $d_{H_0}(q) \geq 2$. However as any edge containing $q$ must also lie in $H_1$ or $H_2$, as $q \notin Y$, we note that $d_{H_0}(q) = 2$. Let $e_q$ be the edge in $H_2$ that contains $q$. Note that $e_q \neq e_2$, by Claim F. As $e_q$ and $e_2$ do not intersect we note that $|N_{H_0}[V(e)]| = 7 \geq 6$, so we are done by Claim D.

*Claim I: Let $e_1 \in E_1$ and $e_2 \in E_2$ be the edges containing $x$ (defined in Definition G). They exist by Claim H. Then $V(e_1) \cap V(e_2) = \{x\}$.*

*Proof of Claim I:* Assume for the sake of contradiction that $|V(e_1) \cap V(e_2)| \geq 2$. If $|V(e_1) \cap V(e_2)| = 3$, then we delete $e_1$ from $H_0$ and add $V(e_1)$ to $Y$. This contradicts the "minimality" of $H$, as $T^*$ remains unchanged, $S^*$ decreases by $1/4$ and $N[Y]$ increases from Claim A by at most 27. Therefore assume that $|V(e_1) \cap V(e_2)| = 2$. Let $e_1 = \{x, v, w\}$ and let $e_2 = \{x, v, y\}$ where $w \neq y$. As $d_{H_0}(x) = d_{H_0}(v) = 2$, there is an edge, $e'$, in $H_0$ such that $\{w, y\} \subseteq V(e')$, by Claim F. However $e' \notin E(H_1)$ and $e' \notin E(H_2)$ by Claim E. This is however a contradiction to (d), as $w, y \notin Y$.

*Claim J: We now obtain a contradiction.*

*Proof of Claim J:* : Let $e_1 \in E_1$ and $e_2 \in E_2$ be the edges containing $x$ (defined in Definition G). They exist by Claim H and $V(e_1) \cap V(e_2) = \{x\}$, by Claim I. Let $e_1 = \{x, v_1, v_2\}$ and let $e_2 = \{x, w_1, w_2\}$. Let $e_1'$ be the edge in $H_1$ containing $w_1$ and let $e_1''$ be the edge in $H_1$ containing $w_2$ (they exist by Claim H). Let $e_2'$ be the edge in $H_2$ containing $v_1$ and let $e_2''$ be the edge in $H_2$ containing $v_2$ (they exist by Claim H).

If $e_1' = e_1''$, then $V(e_1') \cap V(e_2) = \{w_1, w_2\}$ and $e_1' = \{w_1, w_2, r\}$ for some $r \in V(H_0)$. By Claim F, there is an edge in $H_0$ that contains $x$ and $r$. But this is a contradiction, as neither $e_1$ or $e_2$ contain $r$, by Claim H. Therefore $e_1' \neq e_1''$. Analogously we can show that $e_2' \neq e_2''$.

We now delete $e_1, e_1', e_1''$ from $H$, $H_0$ and $H_1$. Delete $e_2, e_2', e_2''$ from $H$, $H_0$ and $H_2$. Delete $V(e_1) \cup V(e_1') \cup V(e_1'')$ from $V(H_1)$ and delete $V(e_2) \cup V(e_2') \cup V(e_2'')$ from $V(H_2)$. Delete $V(e_1) \cup V(e_2)$ from $H$ and $H_0$. Let $S_1$ be any subset of size three in $V(e_1') \cup V(e_1'') - \{w_1, w_2\}$ and let $S_2$ be any subset of size three in $V(e_2') \cup V(e_2'') - \{v_1, v_2\}$. Add the edges $S_1$ and $S_2$ to $H$ and $H_0$. Finally add all vertices in $V(e_1') \cup V(e_1'') \cup V(e_2') \cup V(e_2'') - \{w_1, w_2, v_1, v_2, x\}$ to $Y$.

We first show that $T^*$ decreases by at most 8. It is clear that the transversal size drops by three in both $H_1$ and $H_2$. So assume that $T'$ is a transversal of the new $H_0$. As in the proof of Claim C we note that one of the three edges $e_1, e_2', e_2''$ are already covered by a vertex in $T'$ (due to $S_2$) and the other two edges can be covered by one additional vertex. Similarly by adding one more vertex to $T'$ we can make sure that $e_2, e_1', e_1''$ are all covered. Therefore the transversal size drops by at most two in $H_0$.

Note that $S^*$ drops by $33/4$ as we delete 9 vertices in each of $H_1$ and $H_2$ and we delete 5 vertices in $H_0$. We also delete three edges in each of $H_1$ and $H_2$ and six edges in $H_0$. But we also add two edges in $H_0$.

$N[Y]$ increases by at most 72 vertices by Claim A, as $|V(e_1') \cup V(e_1'') \cup V(e_2') \cup V(e_2'') - \{w_1, w_2, v_1, v_2, x\}| \leq 8$. As $V^*$ decreases by at most 13, we note that $V^* - N[Y]$ decreases by at most 85. We note that (a)-(d) still holds after the above operations. We therefore have a contradiction to the "minimality" of $H$, as $8 \leq 33/4 - 85/372$. $\square$

**Theorem 8** *If $G$ is a graph with $\delta(G) \geq 3$ and two partition classes then $f_t(G) \leq (\frac{5}{4} - \frac{1}{372})|V(G)|$.*

**Proof.** Let $G$ be any graph with $\delta(G) \geq 3$ and let $(W_1, W_2)$ be a partition of $V(G)$. Define the hypergraph $H_G$, such that $V(H_G) = V(G)$ and $E(H_G)$ is obtained by selecting for each $v \in V(G)$ one set of three vertices from $N_G(v)$ to form a hyperedge. $E(H_G) = \{e_v : v \in V(G)\}$, $e_v = \{x_v, y_v, z_v\} \subseteq N_G(v)$. Furthermore for every hyperedge, $e \in E(H_G)$ let $L(e)$ be the set $\{0, i\}$ if $v \in W_i$. For reasons which will be clear later we let $L(v) = \{0, 1, 2\}$ for every $v \in V(H_G)$. Let $H_i$ be the 3-uniform hypergraph containing vertex-set $V_i = \{v : i \in L(v) \text{ and } v \in V(H)\}$ and edge-set $E_i = \{e : i \in L(e) \text{ and } e \in E(H)\}$, for $i = 0, 1, 2$. Note that a transversal of $H_0$ corresponds to a total dominating set in $G$ and a transversal of $H_i$ ($i \in \{1, 2\}$) corresponds to a total dominating set in $G$ of the set $W_i$. Therefore we would be done if we could show that $\mathcal{T}(H_0) + \mathcal{T}(H_1) + \mathcal{T}(H_2) \leq (\frac{5}{4} - \frac{1}{372})|V(G)|$. Let $Y$ be an empty set. We note that $|E_1| + |E_2| = |E_0| = |V_0| = |V_1| = |V_2| = |V(H_0) \cap V(H_1) \cap V(H_2) \backslash N_H[Y]| = |V(G)|$ and therefore the inequality above is equivalent to

$$(*) \qquad \sum_{i=0}^{2} \mathcal{T}(H_i) \leq \left( \sum_{i=0}^{2} \frac{|V_i| + |E_i|}{4} \right) - \frac{|V(H_0) \cap V(H_1) \cap V(H_2) \setminus N_H[Y]|}{372}$$

For simplicity we will use the following notation:

$$
\begin{array}{rcl}
T^* & = & \sum_{i=0}^{2} \mathcal{T}(H_i) \\
S^* & = & \sum_{i=0}^{2} \frac{|V_i|+|E_i|}{4} \\
V^* & = & V(H_0) \cap V(H_1) \cap V(H_2)
\end{array}
$$

We will now do a few transformations on $H, H_0, H_1, H_2$.

*Transformation 1:* While there is some vertex $x \in V(H)$ with $d_{H_0}(x) \geq 5$ (or equivalently $d_H(x) \geq 5$), delete $x$ and all edges incident with $x$ from $H$ (and therefore also from $H_0$, $H_1$ and $H_2$).

*Claim A: If (*) holds for the resulting hypergraphs, then it also holds for our original hypergraphs.*

*Proof of Claim A:* We note that $T^*$ drops by at most three, as we may place $x$ in the transversal of the new $H_i$'s in order to get transversals in the old $H_i$'s. We note that $S^*$ decreases by at least $13/4$, as we delete $x$ from $H_0, H_1, H_2$ and 5 edges from $H_0$ plus a total of 5 edges from $H_1$ and $H_2$. As $V^*$ decreases by one and $N_H[Y] = \emptyset$ remains unchanged, we are done.

*Transformation 2:* While there is a vertex $x \in V(H)$ with $d_{H_1}(x) \geq 3$, delete $x$ and all edges incident to $x$ from $H_0$ and $H_1$. Also delete these edges from $H$ (but do not delete $x$ or any edges incident to $x$ in $H_2$). If $d_{H_2}(x) = 0$ then delete $x$ from $H_2$ (i.e. delete 2 from $L(x)$). If $d_{H_2}(x) > 0$ then note that $d_{H_2}(x) = 1$ (as we have performed transformation 1 as long as we could) and put $N_{H_2}[x]$ in $Y$.

*Claim B: If (*) holds for the resulting hypergraphs, then it also holds for our original hypergraphs.*

*Proof of Claim B:* We note that $T^*$ drops by at most two, as we may place $x$ in the transversal of the new $H_0$ and $H_1$ in order to get transversals in the old $H_0$ and $H_1$. We note that $S^*$ decreases by at least $9/4$, as we delete 3 edges and 1 vertex from $H_0$ and $H_1$ and we either delete a vertex in $H_2$ or 4 edges from $H_0$. As $V^*$ decreases by one and $N_H[Y]$ increases by at most 21 (as $\Delta(H) \leq 4$, after Transformation 1), we are done.

*Transformation 3:* While there is a vertex $x \in V(H)$ with $d_{H_2}(x) \geq 3$, then do the following. Delete $x$ and all edges incident to $x$ from $H_0$ and $H_2$. Also delete these edges from $H$ (but do not delete $x$ or any edges incident to $x$ in $H_1$). Furthermore delete any vertices in $H_2$, which get degree zero by the above transformation. If $d_{H_1}(x) = 0$ then delete $x$ from $H_1$. If $d_{H_1}(x) > 0$, then we put $N_{H_1}[x]$ in $Y$.

*Claim C: If (*) holds for the resulting hypergraphs, then it also holds for our original hypergraphs.*

*Proof of Claim C:* We note that $T^*$ drops by at most two, as we may place $x$ in the transversal of the new $H_0$ and $H_2$ in order to get transversals in the old $H_0$ and $H_2$. Lets count any edge, $e$, in $H_1$, which does not lie in $H_0$ as contributing $\frac{1+|V(e)\cap V(H_0)|/3}{4}$ to the sum $S^*$. We note that there are no such edges when we start the transformation 3's.

We note that $S^*$ now decreases by at least 25/12, because of the following. For every edge containing $x$ in $H_2$, which does not lie in $H_0$ there is a vertex of degree one in the edge, due to the above transformations. Therefore we either delete an edge in $H_0$ or a vertex in $H_2$ for each of the edges containing $x$ in $H_2$. As we also delete the edges in $H_2$ and the vertex $x$ in $H_0$ and $H_2$ we note that $S^*$ drops by at least 8/4. So if $d_{H_1}(x) = 0$ then $S^*$ decreases by at least 9/4 as claimed. If $d_{H_1}(x) > 0$ and the edge, $e$, containing $x$ in $H_1$ also lies in $H_0$, then we are done as we delete an extra edge in $H_0$ and the edge left in $H_1$ is counted as at most $(1 + 2/3)/4$. If $d_{H_1}(x) > 0$ and the edge, $e$, containing $x$ in $H_1$ does not lie in $H_0$, then we decrease the contribute to $S^*$ from $e$ by 1/12 as $|V(e) \cap V(H_0)|$ decreases. This shows that $S^*$ decreases by at least 25/12.

As $V^*$ decreases by one and $N[Y]$ increases by at most 21 (as $\Delta(H) \leq 4$, after Transformation 1), we are done.

*Transformation 4:* If $e_1, e_2 \in E(H_i)$ and $|V(e_1) \cap V(e_2)| \geq 2$ for some $i \in \{1, 2\}$, then we do the following.

If $|V(e_1) \cap V(e_2)| = 3$, then if $e_1, e_2 \in E_0$ we delete $e_2$ from both $H_0$ and $H_i$. If $e_j \notin E_0$ ($j \in \{1, 2\}$) then we delete $e_j$ from $H_i$ (in this case $V(e_j) \subseteq Y$). So now assume that $|V(e_1) \cap V(e_2)| = 2$ and $e_1 = (u_1, x, y)$ and $e_2 = (u_2, x, y)$, where $u_1 \neq u_2$,

If $d_{H_i}(u_1) = d_{H_i}(u_2) = 2$, then by the above transformations we note that $e_1, e_2 \in E_0$. We now add a new vertex $q$ to $H$, $H_0$ and $H_i$. We delete $e_1$ and $e_2$ from $H$, $H_i$ and $H_0$ and add the edges $\{q, x, y\}$ to $H$, $H_i$ and $H_0$.

If $d_{H_i}(u_j) = 1$, for some $j \in \{1, 2\}$, then do the following. Delete $e_1$, $e_2$ and the vertices $\{u_j, x, y\}$ from $H_i$. Add the vertices $\{u_1, u_2, x, y\}$ to $Y$.

*Claim D: If (*) holds for the resulting hypergraphs, then it also holds for our original hypergraphs.*

*Proof of Claim D:* In the case when $|V(e_1) \cap V(e_2)| = 3$ we note that $T^*$ remains unchanged, $S^*$ decreases by 1/4 and $V^* - N[Y]$ remains unchanged.

We are now done with this case.

In the case when $d_{H_i}(u_1) = d_{H_i}(u_2) = 2$, we note that $T^*$, $S^*$ and $V^*$ remain unchanged and $N[Y]$ can only grow by adding $q$ to it, but $q \notin V^*$. We also note that the above transformation decreases the number of edges in $H_i$, so it cannot continue indefinitely. We are now done with this case.

In the case when $d_{H_i}(u_j) = 1$, we note that $T^*$ decreases by at most one, $S^*$ decreases by 5/4, $V^*$ decreases by at most three and $N[Y]$ increases by at most 24 (In $H - e_1 - e_2$ we note that $u_1$ and $u_2$ have degree at most 3 while $x$ and $y$ have degree at most 2). As $1/4 \geq 27/372$ we are done with this case.

*Claim E: $\Delta(H_1), \Delta(H_2) \leq 2$ and $\Delta(H - E(Y)) \leq 4$ and there are no overlapping edges in $H_i$, $i \in \{1, 2\}$.*

*Proof of Claim E:* The fact that $\Delta(H_1), \Delta(H_2) \leq 2$ follow from Transformations 2 and 3. As $\Delta(H) \leq 4$ after Transformation 1 and no other transformation increases $\Delta(H)$, we note that $\Delta(H - E(Y)) \leq \Delta(H) \leq 4$. There are no overlapping edges in $H_i$, $i \in \{1, 2\}$ due to Transformation 4.

*Claim F: If $e \in E(H) - E(Y)$, then $0 \in L(e)$ and $|L(e)| \geq 2$.*

*Proof of Claim F:* This was true before Transformation 1 as it was true for all edges. Transformation 1 clearly does not change this property. In Transformation 2, we only keep an edge, $e$, in $H_i$, where $i \in \{1, 2\}$ but delete it in $H_0$ if we put $V(e)$ in $Y$. So the above still holds after Transformation 2. Analogously it also holds after Transformation 3. It is not difficult to check that it also holds after Transformation 4 (note that the above property holds for the edge we might add to $H$ in Transformation 4).

We now see that (*) holds due to Lemma 3. That implies the theorem. □

# 8 Possible strengthening of Theorem 8

No graph extremal for Theorem 8 is known and probably an inequality $f_t(G) \leq \alpha|V(G)|$ can be obtained for some $\alpha$ smaller than $\frac{5}{4} - \frac{1}{372}$. Certainly $\alpha$ must be at least 9/8, that is demonstrated by the graphs of section 6.

There is a graph of order 12 having $f_t(H_{12}) = 7n/6$, namely $H_{12}$ from the family $\mathcal{H}$ defined after Theorem 2, with the two $P_6$'s as its partition classes. Unless we, e.g., demand that the order of the graphs be large, $H_{12}$ shows that we cannot get a better inequality than the following conjecture.

**Conjecture 1** *Let $G$ be a graph of order $n$ with $\delta \geq 3$ and let $V(G)$ be partitioned into two classes. Then $f_t(G) \leq 7n/6$.*

## 9 Three partition classes

**Theorem 9** *Let $G$ be a graph of order $n$ with $\delta \geq 3$ and let $V(G)$ be partitioned into three classes. Then $f_t(G) \leq 3n/2$.*

*For arbitrarily large $n$, $n \equiv 0 \pmod 6$, there exist graphs $G_n$ with $g_t(G_n) = n$, $\gamma_t(G_n) = n/3$, $f_t(G) = 4n/3$.*

**Proof.** By Theorem 1 we have that $\gamma_t(G) \leq n/2$, and $g_t(G) \leq n$ holds trivially, so by addition we get $f_t(G) \leq 3n/2$ as desired.

Assume a graph $G$ has $g_t(G) = n$. Then $\Delta(G) \leq 3$ and as $\delta(G) \geq 3$, $G$ is cubic. Since each vertex has three neighbours, one in each partition class, we see for each $i = 1, 2, 3$, that vertices in class $V_i$ span a matching in $G$.

Listing the 3 neighbours to each $V_i$-vertex we count each vertex of $G$ once, so $3|V_i| = n$ giving $|V_1| = |V_2| = |V_3| = n/3$.

Each $V_1$-vertex is adjacent to precisely one $V_2$-vertex and that has no other $V_1$-neighbour, so there is a perfect matching of $V_1V_2$-edges and analogously $G$ contains perfect matchings of $V_1V_3$- and $V_2V_3$-edges.

One partition class $V_i$ totally dominates $G$ so $\gamma_t(G) \leq n/3$. In fact, $\gamma_t(G) = n/3$ because each vertex in $G$ can totally dominate at most its three neighbours.

Following the steps above, it is now easy for $n \equiv 0 \pmod 6$ to construct a graph $G_n$ with $g_t(G_n) = n$. This graph has $f_t(G_n) = \gamma_t(G_n) + g_t(G_n) = 4n/3$.
□

We do not know if there, for $k = 3, \delta \geq 3$, are graphs $G$ with $4n/3 < f_t(G) \leq 3n/2$, but we pose the following conjecture.

**Conjecture 2** *There exists some positive $\epsilon$ such that the following holds. If $G$ is a graph with $\delta(G) \geq 3$ and the vertices of $G$ are partitioned into three classes, then $f_t(G) \leq (3/2 - \epsilon)|V(G)|$.*

**Theorem 10** *Let $G$ be a graph of order $n$ with $\delta \geq 3$, partitioned into at least four classes. Then $f_t(G) \leq 3n/2$ and there exists an infinite family of graphs with $f_t(G) = 3n/2$.*

**Proof.** The inequality is proven as in Theorem 9. For a graph with $f_t(H) = 3n/2$ take $H \in \mathcal{H}$ ($\mathcal{H}$ is defined after Theorem 2). Let $v_1, v_2, \ldots, v_{n/2}$ and $u_1, u_2, \ldots, u_{n/2}$ be two disjoint paths in $H$ such that $\{v_1u_2, v_2u_1, v_1v_{n/2}, u_1u_{n/2}\} \subseteq E(H)$. Let $V_1, V_2, V_3, V_4$ be a partition of $H$ such that $l(v_1), l(v_2), \ldots, l(v_{n/2})\text{....} = 1, 2, 3, 4, 1, 2, 3, 4, \text{.....}$ and $l(u_1), l(u_2), \ldots, l(u_{n/2})\text{....} = 4, 3, 2, 1, 4, 3, 2, 1, \text{.....}$ where $l(x) = i$ if $x \in V_i$, then $f_t(H; V_1, V_2, V_3, V_4) = 3n/2$. $\square$

# References


[1] G. Chartrand and L. Lesniak, *Graphs and Digraphs: Third Edition*, Chapman & Hall, London, 1996.

[2] V. Chvátal and C. McDiarmid, Small transversals in hypergraphs. *Combinatorica* **12** (1992), 19–26.

[3] O. Favaron, M.A. Henning, C.M. Mynhardt and J. Puech, Total domination in graphs with minimum degree three, *J. of Graph Theory* **34(1)** *(2000), 9-19.*

[4] A. Frendrup, M. A. Henning and P.D. Vestergaard, Total domination in partitioned trees and partitioned graphs with minimum degree two, pp. 1-19, subm. to *Journal of Global Optimization.*

[5] T.W. Haynes, S.T. Hedetniemi, and P.J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, New York, 1998.

[6] T.W. Haynes, S.T. Hedetniemi, and P.J. Slater (eds.), *Domination in Graphs: Advanced Topics*, Marcel Dekker, New York, 1998.

[7] B.L. Hartnell and P.D. Vestergaard, Partitions and dominations in a graph. *J. Combin. Math. Combin. Comput.* **46** (2003), 113–128.

[8] M.A. Henning, Graphs with large total domination number. *J. Graph Theory* **35(1)** (2000), 21–45.

[9] M.A. Henning, Recent results on total domination in graphs: A survey. pp. 1-26. *Manuscript.*

[10] M.A. Henning, L. Kang, E. Shan and A. Yeo, On matching and total domination in graphs, pp. 1-8, *Manuscript.*

[11] M.A. Henning and P.D. Vestergaard,“Domination in partitioned graphs with minimum degree two”’, pp. 1-29. To appear in *Discrete Mathematics.*

[12] M.A. Henning and A. Yeo, Hypergraphs with large transversal number and with edgesize at least three, pp. 1-22. *Manuscript.*

[13] M. A. Henning and A. Yeo, Tight lower bounds on the size of a matching in a regular graph. *submitted.*

[14] S.M. Seager, Partition dominations of graphs of minimum degree 2. *Congress. Numer.* **132** (1998), 85–91.

[15] S. Thomassé and A. Yeo, Total domination of graphs and small transversals of hypergraphs. To appear in *Combinatorica.*

[16] Z. Tuza, Covering all cliques of a graph. *Discrete Math.* **86** (1990), 117–126.

[17] Z. Tuza and P.D. Vestergaard, Domination in partitioned graph. *Discussiones Mathematicae Graph Theory.* **22**(1) (2002), 199–210.

[18] A. Yeo, Relationships between total domination, order, size and maximum degree of graphs, pp. 1-12. *Manuscript.*

[19] A. Yeo, Excluding one graph significantly improves bounds on total domination in connected graphs of minimum degree four. *In preperation.*

# Distance Domination in Partitioned Graphs

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**Abstract**

We treat a variation of domination which involves a partition $(V_1, V_2, \ldots, V_k)$ of $V(G)$ and domination of each partition class $V_i$ over distance $d$ where all vertices and edges of $G$ may be used in the domination process. Strict upper bounds and extremal graphs are presented, the results are collected in three handy tables. Further, we compare a high number of partition classes and the number of dominators needed.

**Keywords:** domination; distance; partition; domination number; partitioned graph
**AMS subject classification: 05C69**

## 1 Notation

By $G = (V, E)$ we denote a *graph* $G$ with *vertex* set $V = V(G)$ and *edge* set $E = E(G)$. The *order* of $G$ is $|V(G)| = n$ and the *size* of $G$ is $|E(G)|$. For $x \in V(G)$ we denote by $N_G(x)$ the set of *neighbours* to $x$ and $N_G[x] = \{x\} \cup N_G(x)$. Analogously for a subset $D$ of $V(G)$ we define $N_G(D) = \cup\{N_G(x) \mid x \in D\}$ and $N_G[D] = \cup\{N_G[x] \mid x \in D\} = D \cup N_G(D)$. Let $d$ be a positive integer, we let $N_{(d,G)}(x)$ denote the set of vertices in $V \setminus \{x\}$ having distance at most $d$ to $x$, and we define $N_{(d,G)}[x] = N_{(d,G)}(x) \cup \{x\}$. We let $N_{(d,G)}(D) = \cup\{N_{(d,G)}(x) \mid x \in D\}$ and $N_{(d,G)}[D] = \cup\{N_{(d,G)}[x] \mid x \in D\}$. Indices may be omitted if clear from context. The *degree* of $x$ is $d_G(x) = |N_G(x)|$, the number of neighbours to $x$. We let $\delta(G) = \delta$ denote the *minimum degree* in $G$ and $\Delta(G) = \Delta$ the *maximum* degree. A *corona graph* $G$, denoted by $G = H \circ K_1$, has order $2n$ and is obtained

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from a graph $H$ of order $n$ and $n$ new vertices, one corresponding to each vertex of $H$, by joining each vertex of $H$ to its corresponding new vertex. Analogously $G = H \circ P_d$ denotes a *$P_d$-corona graph* $G$ of order $n(d+1)$ obtained as the disjoint union of a graph $H$ of order $n$ and $n$ disjoint paths $P_d$, each of length $d-1$, by joining each vertex of $H$ to an end vertex of its corresponding path $P_d$. In $G$ a set $S$ of vertices is called *distance $d$ independent* if the distance between any two vertices of $S$ is at least $d+1$. For $S \subseteq V(G)$ we denote by $G[S]$ the subgraph of $G$ spanned by $S$.

A set $S, S \subseteq V$, in a graph $G$ *dominates* $G$ if every vertex in $G \setminus S$ is adjacent to some vertex of $S$. The minimum number of vertices needed to dominate $V$ is the *domination number* $\gamma(G)$.

A set $S, S \subseteq V$, in a graph $G$ *distance $d$ dominates* $G$ if every vertex in $G \setminus S$ has distance at most $d$ to some vertex of $S$, i.e. if $V \subseteq \bigcup_{x \in S} N_d[x]$

The minimum number of vertices needed to distance $d$ dominate $V$ is the *distance $d$ domination number* $\gamma_d(G)$. For $d = 1$ we have the ordinary domination, $\gamma_1(G) = \gamma(G)$.

For $V_i \subseteq V$ we define $\gamma_d(G; V_i)$ to be the minimum cardinality of a set $S$, $S \subseteq V$, such that each vertex $v \in V_i \setminus S$ satisfy that $N_{(d,G)[v]} \cap S \neq \emptyset$.

A *partition* $V = (V_1, V_2, \ldots, V_k)$ of $V(G)$ into $k$ disjoint sets, $k \geq 2$, has $V = \bigcup_{i=1}^{k} V_i$, $V_i \cap V_j = \emptyset$, $1 \leq i < j \leq k$. For an integer $k \geq 1$ and a partition $(V_1, V_2, \ldots, V_k)$ of $V$, we define for distance $d = 1$ the following:

$$
\begin{aligned}
&f(G; V_1, V_2, \ldots, V_k) = \gamma(G) + \gamma(G; V_1) + \gamma(G; V_2) + \ldots + \gamma(G; V_k) \\
&g(G; V_1, V_2, \ldots, V_k) = \gamma(G; V_1) + \gamma(G; V_2) + \ldots + \gamma(G; V_k) \\
&f(k, G) = \max\{f(G; V_1, V_2, \ldots, V_k) \mid V_1, V_2, \ldots, V_k \text{ is a partition of } V\} \\
&g(k, G) = \max\{g(G; V_1, V_2, \ldots, V_k) \mid V_1, V_2, \ldots, V_k \text{ is a partition of } V\}
\end{aligned}
$$

We observe that $f(k, G) = \gamma(G) + g(k, G)$. For distance at most $d$, $d \geq 1$, definitions of $f_d(G; V_1, V_2, \ldots, V_k)$ etc. are analogous. For further notation we refer to Chartrand and Lesniak [5].

## 2 Introduction

Half a century ago Ore [15] defined domination and proved that a connected graph $G$ of order $n$ has $\gamma(G) \leq n/2$ . Payan and Xuong [17] proved that equality, $\gamma(G) = n/2$, holds precisely for $C_4$ and corona graphs. Fink, Jacobson, Kinch and Roberts also provided a proof [6]. Obviously, for connected graphs of fixed order $n$ their domination number will decrease with increasing size as illustrated in Table 1.

| Bounds for $\gamma(G)$ when $G$ is connected and has order $n$. | | |
|---|---|---|
| $\delta(G)$ | Bound | Reference |
| $\delta \geq 1$ | $\gamma(G) \leq n/2$ | [15] |
| $\delta \geq 1$ | $\gamma(G) \leq \frac{n+2-\delta(G)}{2}$ | [16] |
| $\delta \geq 2$<br>$n \geq 8$ | $\gamma(G) \leq 2n/5$ | [13] |
| $\delta \geq 3$ | $\gamma(G) \leq 3n/8$ | [18] |
| $\delta \geq 1$ | $\gamma(G) \leq \frac{1+\ln(\delta(G)+1)}{\delta(G)+1} \cdot n$ | [1], |
| $\delta \geq 1$ | $\gamma(G) \leq \left(1 - \delta(G)(\frac{1}{\delta(G)+1})^{1+\frac{1}{\delta}}\right) \cdot n$ | [2], [16]<br>[3, 4] |
| $\delta \geq 1$ | $\gamma(G) \leq \frac{n}{\delta(G)+1} \sum_{j=1}^{\delta(G)+1} \frac{1}{j}$ | [2], [16] |

Several variants of domination in graphs have been surveyed in two books by Haynes, Hedeniemi and Slater [9, 10]. We shall here be concerned with distance domination in partitioned graphs.

A graph $G$ has its various domination numbers bounded above by the corresponding domination number for any one of its spanning trees $T$, e.g. $f(2, G) \leq f(2, T)$, and if we search for an upper bound holding for all connected graphs of order $n$ it suffices to search among all trees of order $n$, e.g. $f(2, G) \leq f(2, T) \leq \frac{5n}{4}$. In the following table $T$ is a tree of order $n$ and $G$ is a connected graph.

| Bounds for $f$ and $g$ for 2-partitioned graphs | | |
|---|---|---|
| Bound | Assumptions | Reference |
| $f(2, T) \leq \frac{5}{4} \cdot n$ | $d = 1, n \geq 3$ | [8] |
| $g(2, T) \leq \frac{4}{5} \cdot n$ | $d = 1, n \geq 3$ | [20] |
| $f(2, G) \leq n$ | $d = 1, \delta \geq 2$ | [19] |
| $g(2, G) \leq \frac{2}{3} \cdot n$ | $d = 1, \delta \geq 2$ | [20] |
| $g(2, G) \leq \frac{\delta+1}{2\delta} \cdot n$ | $d = 1, \delta \geq 1$ | [20] |
| $f_d(2, T) \leq \frac{6}{2d+3} \cdot n$ | $d \geq 2, n \geq d+2$ | [7] |
| $g_d(2, T) \leq \frac{4}{2d+3} \cdot n$ | $d \geq 2, n \geq d+2$ | |

| Bounds for $f$ and $g$ for 3-partitioned graphs | | |
|---|---|---|
| Bound | Assumptions | Reference |
| $f(3,T) \leq \frac{7}{5} \cdot n$ | $d = 1, n \geq 3$ | [8] |
| $f_2(3,T) \leq n$ | $d = 2, n \geq 4$ | [7] |
| $f_2(3,T) \leq \frac{30}{31} \cdot n$ | $d = 2, n \geq 5$ $T \notin \{P_6, P_7, P_8, G_{10}\}$ | Theorem 2 |
| $g_2(3,T) \leq \frac{18}{25} \cdot n$ | $d = 2, n \geq 5$ $T \neq T_8$ | Theorem 2 |
| $f_3(3,T) \leq \frac{24}{31} \cdot n$ | $d = 3, n \geq 6,$ $T \notin \{P_9, P_{10}\}$ | Theorem 2 |
| $f_d(3,T) \leq \frac{24}{6d+13} \cdot n$ | $d \geq 4, n \geq d+3,$ $T \neq P_{2d+4}$ | Theorem 2 |
| $g_d(3,T) \leq \frac{18}{6d+13} \cdot n$ | $d \geq 3, n \geq d+3$ $T \neq P_{2d+4}$ | Theorem 2 |

Ore's theorem that $\gamma(G) \leq n/2$ generalises from $d = 1$ to $d \geq 1$.

**Theorem 1.** Let $d \geq 1$ be an integer and let $G$ be a connected graph of order $n \geq d+1$then $\gamma_d(G) \leq \frac{n}{d+1}$ where equality holds if and only if $n = d+1$, $G \cong C_{2d+2}$ or $G \cong H \circ P_d$ for a connected graph $H$.

*Proof.* A connected graph $G$ with diameter less than $d$ has $\gamma_d(G) = 1$ and $1 = \gamma_d(G) = \frac{n}{d+1}$ can occur for any connected graph with $n = d+1$. So assume $G$ is connected and has diameter at least $d$. The inequality was proven by induction by Henning, Oellermann and Swart [11]. We give a short, direct proof. Choose $v$ to be an end vertex of a diametrical path in a spanning tree $T$ for $G$. Consider distance classes from $v$

$$A_i = \{x \in V(T) | d_T(x,v) = i\}, i = 0, 1, 2, \ldots$$

Each of the $d+1$ sets $B_i = \bigcup_{j=0}^{\infty} A_{i+j(d+1)}, i = 0, 1, \ldots, d$, distance $d$ dominates $G$ and as $\bigcup_{j=0}^{d} B_i = V(T)$ we can among the $d+1$ sets choose a smallest set $B_{i_0}$ and obtain $\gamma_d(G) \leq |B_{i_0}| \leq \frac{n}{d+1}$.
M.A. Henning [10, Prop. 12.4] proves that there exists a minimum distance d dominating set $D$ such that each vertex $x \in D$ has a *private neighbour* $x'$ at distance exactly $d$ from $x$, i.e.,

$$\forall x \in D \exists x' \in V(G) \setminus D : N_d(x') \cap D = \{x\} \text{ and } d_G\{x, x'\} = d.$$

Assume next that $\gamma_d(G) = \frac{n}{d+1}$ and let $D$ be such a minimum distance d dominating set with $|D| = \frac{n}{d+1}$. Let $D'$ denote this set of distance $d$ private neighbours to $D$,

$$D' = \{x' \in V(G) | \exists x \in D : N_d(x') \cap D = \{x\} \text{ and } d_G\{x, x'\} = d\}.$$

Thus from each $x \in D$ emanates a path $P_x : x_0x_1x_2, \ldots, x_d = x'$ where $x' \in D'$. If $x, y \in D, x \neq y$, we have $V(P_x) \cap V(P_y) = \emptyset$, as $x', y'$ are private neighbours to $x, y$, resp. Since $|D| = \frac{n}{d+1}$ and each path $P_x, x \in D$, has $d+1$ vertices we see that $V(G) = \bigcup_{x \in D} V(P_x)$ and $|D| = |D'| = \frac{n}{d+1}$. For $|D| = |D'| = 2$ the circuit $G = C_{2d+2}$ may occur, but for $|D| \neq 2$ every edge either is on a path $P_x$ or has both its ends in precisely one of the sets $D$ or $D'$. That implies by connectivity of $G$ that one of the sets $D$ or $D'$, say $D'$, is independent. Consequently $G = H \circ P_d$, where $V(H) = D$. □

From Theorem 1 we immediately obtain

**Observation 1.** If $G$ is a graph and $k, d \geq 1$ are integers then $g_d(k, T) \leq |V(G)|$ and if $G$ is a connected graph such that $|V(G)| \geq d+1$ then $f_d(k, G) \leq \frac{d+2}{d+1}|V(G)|$.

# 3 Bounds for $f_d(3, T)$ and $g_d(3, T)$

In the following we prove optimal bounds for $f_d(3, T)$ and $g_d(3, T)$ when $T$ is a tree with at least $d+2$ vertices. First some families of graphs are defined.

For each integer $d \geq 2$ let $Q_d$ be the family of trees consisting of $P_{2d+4}$ and all trees with $d+2$ vertices. Let $G_{10}$ denote $P_9$ with a pendent vertex attached to its center, i.e, the graph with 10 vertices illustrated in figure 1.

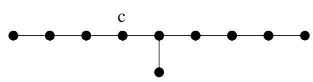


Figure 1: Illustration of the graph $G_{10}$.

A neighbour $c$ to the center of the path $v_1, v_2, v_3, c, v_5, \ldots, v_9$ in $G_{10}$ is called a connection-vertex in $G_{10}$. Let $Q'_2 = Q_2 \cup \{P_6, P_7, G_{10}\}$, $Q'_3 = Q_3 \cup \{P_9\}$ and let $Q'_d = Q_d$ for $d \geq 4$.

For $d \geq 2$ let $T_d$ be the tree with the smallest diameter that can be obtained from $3P_{2d+4} \cup K_1$ by adding three edges all incident with the isolated vertex which will be called the central vertex in $T_d$. For $d \geq 2$ we define $\mathcal{F}_d$ as the family of trees that can be obtained from graphs isomorphic to $T_d$ by adding edges between their central vertices. Let $T'_2$ be the tree obtained from $3G_{10} \cup K_1$ by adding three edges all incident with the isolated vertex (this vertex will be called central in $T_2'$) and a connection-vertex from each of the three $G_{10}$-components. Define $\mathcal{F}_d' = \mathcal{F}_d$ for $d \geq 3$ and $\mathcal{F}_2'$ as the family of trees that can be obtained from isomorphic copies of $T_2'$ by adding edges between central vertices.

**Lemma 1.** Let $d \geq 2$ be an integer and let $T$ be a tree with $n \geq d+2$ vertices such that for each edge $e \in E(T)$ a component of $T - e$ has less than $d+2$ vertices. Then

- $g_d(3,T) = \frac{3}{d+2}n$ if $T \in Q_d$ and if $T \notin Q_d$ then $g_d(3,T) < \frac{18}{6d+13}n$.
- If $d = 2$ then $f_d(3,T) = n$ if $T \in Q_2'$ and if $T \notin Q_2'$ then $f_d(3,T) < \frac{30}{31}n$.
- If $d \geq 3$ then $f_d(3,T) = \frac{4}{d+2}n$ if $T \in Q_d$ and if $T \notin Q_d'$ then $f_d(3,T) < \frac{24}{6d+13}n$.

*Proof.* Let $(V_1, V_2, V_3)$ be a partition of $V(T)$ such that $g_d(3,T) = \sum_{i=1}^{3} \gamma_d(V_i)$ and let $P : v_1, \ldots, v_{diam(T)+1}$ be a diametrical path in $T$. By the assumptions for $T$ we have that $diam(T) \leq 2d+2$. In the following we consider three cases.

*diam*$(T) \leq 2d$**:** Since $\gamma_d(T) = 1$ in this case $g_d(3,T) = 3 \leq \frac{3}{d+2}n$ and $f_d(3,T) = 4 \leq \frac{4}{d+2}n$. Equality holds if and only if $n = d+2$. If $n > d+2$ then $g_d(3,T) = 3 < \frac{18}{6d+13}n$ and $f_d(3,T) = 4 < \frac{24}{6d+13}n < \frac{30}{31}n$.

*diam*$(T) = 2d+1$**:** Here it can be assumed that the vertices $v_{d+2}, \ldots, v_{2d+2}$ only are adjacent to vertices from $V(P)$. Hence $\{v_{d+1}, v_{d+2}\}$ is a distance $d$ dominating set for $T$ and if $v_{2d+2} \notin V_i$ then $\{v_{d+1}\}$ is a distance $d$ dominating set for $V_i$. Thus $g_d(3,T) = 4 \leq \frac{4}{2d+2}n < \frac{18}{6d+13}n$ and if $d \geq 3$ then $f_d(3,T) = 6 \leq \frac{6}{2d+2}n < \frac{24}{6d+13}n$. If $d = 2$ then $f_d(3,T) = n$ if $n = 6$ and otherwise $f_d(3,T) = 6 \leq \frac{6}{7}n < \frac{30}{31}n$.

*diam*$(T) = 2d+2$**:** Here we can assume that the vertices from $V(P)\backslash\{v_{d+2}\}$ only are adjacent to vertices from $V(P)$. Let $U$ be the vertices in $T$ at distance $d+1$ from $v_{d+2}$. Let $D_i := (V_i \cap U) \cup \{v_{d+2}\}$ then $D_i$ is a distance $d$ dominating set for $V_i$. Let $D' := \{v_{d+1}, v_{d+3}\} \cup (U\backslash\{v_1, v_{2d+3}\})$ and let $D := D'$ if no end vertex in $T$ has distance $d$ to $v_{d+2}$ and otherwise let $D := D' \cup \{v_{d+2}\}$. Then $D$ distance $d$ dominates $T$ and we obtain that $g_d(3,T) \leq \sum_{i=1}^{3} |D_i| \leq 5 + \frac{n-(2d+3)}{d+1}$ and $f_d(3,T) \leq 7 + 2\frac{n-(2d+3)}{d+1}$.

For $d \geq 2$ we have $5 < \frac{18}{6d+13}(2d+3)$ which together with $\frac{1}{d+1} < \frac{18}{6d+13}$ gives

$$g_d(3,T) < \frac{18}{6d+13}(2d+3) + \frac{18}{6d+13}(n-(2d+3)) = \frac{18}{6d+13}n, (d \geq 2).$$

Analogously for $d \geq 4$ we have $7 < \frac{24}{6d+13}(2d+3)$ and $\frac{1}{d+1} < \frac{24}{6d+13}$ giving

$$f_d(3,T) < \frac{24}{6d+13}(2d+3) + \frac{24}{6d+13}(n-(2d+3)) = \frac{24}{6d+13}n, (d \geq 4).$$

If $d \in \{2,3\}$ and $n > 2d+3$ then

$$\begin{aligned} f_d(3,T) &\leq \left(\frac{7}{2d+3}(2d+3) + 2\frac{1}{d+1}\right) + 2\frac{n-(2d+4)}{d+1} \\ &< \frac{24}{6d+13}(2d+4) + \frac{24}{6d+13}(n-(2d+4)) = \frac{24}{6d+13}n. \end{aligned}$$

□

**Remark** For $d \geq 4$ we have that $Q_d = Q'_d$ and the last statement of Lemma 1 includes all trees of order $n \geq d+2$. For $d = 3$, however, $Q'_d = Q_3 \cup \{P_9\}$, and $P_9$, where $f_3(3, P_9) = 8$, is not included in the statement since $8 > \frac{4}{d+2}n$ (third bullet) and consequently $f_3(3, P_9)$ will not be included in the statement of Theorem 2 below (second bullet).

**Observation 2.** If $G$ is a graph from $Q'_d$ and $(V_1, V_2, V_3)$ is a partition such that $g_d(3,G) = \sum_{i=1}^{3} \gamma_d(G; V_i)$ then for each vertex $v \in V(G)$ there exists an index $i \in \{1,2,3\}$ such that $\gamma_d(G; V_i) = \gamma_d(G; V_i - N_{d-1}[v]) + 1$ and a vertex $x \in N[v]$ such that $x$ is contained in a $\gamma_d(G)$-set. Further each vertex in $V(G_{10})$ is contained in a $\gamma_d(G_{10})$-set.

**Lemma 2.** Let $d \geq 2$ be an integer. If $G_1$ and $G_2$ are contained in $Q'_d$ and $G$ is a connected graph that can be obtained from $G_1 \cup G_2$ by adding an edge, but $G \notin Q'_d$, then $f_d(3,G) < f_d(3,G_1) + f_d(3,G_2)$.

*Proof.* Assume that there exists a graph $G$ obtained by adding an edge $uv$ between two graphs $G_1$ ($u \in V(G_1)$) and $G_2$ ($v \in V(G_2)$) from $Q'_d$ such that $f_d(3,G) = f_d(3,G_1) + f_d(3,G_2)$. Let $(V_1,V_2,V_3)$ be a partition of $G$ such that $f_d(3,G) = \gamma_d(G) + \sum_{i=1}^{3} \gamma_d(G; V_i)$. It follows by the assumptions that $\gamma_d(G_1; V_i \cap V(G_1)) + \gamma_d(G_2; V_i \cap V(G_2)) = \gamma_d(G; V_i)$ for $i \in \{1,2,3\}$. Assume that $G_1 \in Q_d$. If $G_1 \ncong P_{d+2}$ or $u$ is not an end vertex then $u$ is in a $\gamma_d(G_1; V_i \cap V(G_1))$-set in $G_1$ for $i \in \{1,2,3\}$. Thus by Observation 2 there exists an index $i \in \{1,2,3\}$ such that $\gamma_d(G; V_i) \leq \gamma_d(G_1; V_i \cap V(G_1)) + \gamma_d(G_2; V_i \cap V(G_2)) - 1$ which gives a contradiction. Thus it can be assumed that $G_1 \notin Q_d$ (and $G_2 \notin Q_d$). Thus the theorem has been proven for $d \geq 4$ since $Q_d = Q'_d$ when $d \geq 4$. In the case where $d \in \{2,3\}$ the lemma can easily be verified for the graphs $G$ obtained when $G_1$ and $G_2$ belongs to $Q'_d \backslash Q_d$ by examining some graphs. □

By considering graphs obtained by adding an edge between to disjoint graphs from $\mathcal{F}'_d$ or $\mathcal{F}_2$ we obtain the following observation.

**Observation 3.** Let $d \geq 2$ and let $\mathcal{F} = \mathcal{F}'_d$ or $\mathcal{F} = \mathcal{F}_2$. If $\{G_1, G_2\} \subset \mathcal{F}$ and $G = G_1 \cup G_2 + uv$ where $u \in V(G_1)$ and $v \in V(G_2)$. Then $G \in \mathcal{F}$ if $f_d(3, G) = f_d(3, G_1) + f_d(3, G_2)$.

**Theorem 2.** Let $d \geq 2$ be an integer and let $T$ be a tree with $n \geq d+2$ vertices then

- If $d = 2$ then $f_d(3, T) = n$ if $T \in Q'_d$ and if $T \notin Q'_d$ then $f_d(3, T) \leq \frac{30}{31}n$ where equality holds if and only if $T \in \mathcal{F}'_d$.
- If $d \geq 3$ then $f_d(3, T) = \frac{4}{d+2}n$ if $T \in Q_d$ and if $T \notin Q'_d$ then $f_d(3, T) \leq \frac{24}{6d+13}n$ and equality holds if and only if $T \in \mathcal{F}'_d$.
- For all $d \geq 2$: $g_d(3, T) = \frac{3}{d+2}n$ if $T \in Q_d$ and if $T \notin Q_d$ then $g_d(3, T) \leq \frac{18}{6d+13}n$ and equality holds if and only if $T \in \mathcal{F}_d$.

*Proof.* The theorem is proven by induction on $n$. The theorem follows from Lemma 1 if we are in the case where $T$ is a tree such that $T - e$ has a component with at most $d + 1$ vertices for each edge $e \in E(T)$.

Thus it can be assumed that there exists an edge $e \in E(T)$ such that both components of $T - e$ have at least $d + 2$ vertices. Let $E_0$ be the set of edges having this property.

*Case 1.* If there exists an edge $e \in E_0$ such that neither of the components $T_1$ and $T_2$ in $T - e$ is contained in $Q'_d$ then for $i \in \{1, 2\}$ the induction hypothesis gives that

$$f_d(3, T_i) \leq \begin{cases} \frac{24}{6d+13}|V(T_i)| & \text{if } d \geq 3 \\ \frac{30}{31}|V(T_i)| & \text{if } d = 2 \end{cases} \tag{1}$$

and

$$g_d(3, T_i) \leq \frac{18}{25}|V(T_i)| \quad \text{if } d = 2. \tag{2}$$

Further, equality holds in (1) if and only if $T_i \in \mathcal{F}'_d$ and equality holds in (2) if and only if $T_i \in \mathcal{F}_2$.

It follows that

$$f_d(3, T) \leq f_d(3, T_1) + f_d(3, T_2) \leq \begin{cases} \frac{24}{6d+13}n & \text{if } d \geq 3 \\ \frac{30}{31}n & \text{if } d = 2 \end{cases} \tag{3}$$

and

$$g_d(3,T) \leq g_d(3,T_1) + g_d(3,T_2) \leq \frac{18}{25}|V(T)| \quad \text{if } d=2. \tag{4}$$

and if equality holds in (3) it follows from Observation 3 that $T \in \mathcal{F}'_d$ and if $d=2$ and equality holds in (4) then analogously $T \in \mathcal{F}_2$.

*Case 2.* If there exists an edge $e \in E_0$ such that both $(T-e)$-components $T_1$ and $T_2$ are in $Q'_d$ then $n \leq 4d+8$ if $d \geq 3$ and $n \leq 20$ if $d=2$. If $d=3$ and $\{T_1,T_2\} \cap \{P_9\} \neq \emptyset$. Since Theorem 2 is easily verified if $T \in Q'_d$ we may assume that $T \notin Q'_d$ and Lemma 2 then implies that $f_d(3,T) \leq f_d(3,T_1) + f_d(3,T_2) - 1$ and calculations give that

$$f_d(3,T) \leq \begin{cases} \frac{7}{4d+8}n < \frac{24}{6d+13}n & \text{if } d \geq 3. \\ \frac{19}{20}n < \frac{30}{31}n & \text{if } d=2, \end{cases}$$

If $d=2$ then $g_d(3,T) \leq \frac{3}{4} f_d(3,T) \leq \frac{57}{80}n < \frac{18}{25}n$.

*Case 3.* Thus it can be assumed that for each edge $e \in E_0$ exactly one of the two $(T-e)$-components belongs to $Q'_d$. Let $\vec{T}$ be the directed graph such that $V(T_d) = V(T)$ and the arcs of $\vec{T}$ are

$$A(\vec{T}) = \{\overrightarrow{uv} | uv \in E_0 \text{ and the component of } T-uv \text{ containing } u \text{ is in } Q'_d\}.$$

Since $T$ is a tree and $E_0 \neq \emptyset$ it follows by taking a longest directed path of $E_0$-arcs in $\overrightarrow{T}$ that there must exist a vertex $x \in \overrightarrow{T}$ with in-degree at least one and out-degree zero. An edge $e$ from $E \setminus E_0$ incident with $x$ has a component of $T-e$ with at most $d+1$ vertices. From these observations we see that each component in $T-x$ has at most $d+1$ vertices or is in $Q'_d$. Further, $T-x$ must contain a component from $Q'_d$. Let $deg_{E_0}(x)$ be the number of components of $T-x$ contained in $Q'_d$ and let $H$ be the induced subgraph of $T$ containing $x$ and the vertices from these components.

Let $(V_1,V_2,V_3)$ be a partition of $V(T)$. From Observation 2 it can be seen that there exists sets $D'_i$ and $D'$ such that

- $\sum_{i=1}^{3} |D'_i| = 3 - deg_{E_0}(x) + g_d(3, H-x)$
- $V(H) \cap V_i \subseteq N_{(d,G)}[D'_i]$, $V(H) \subseteq N_{(d,G)}[D']$, $|D'| = \gamma_d(H-x)$, $x \in D'_i$ and $N_{(2,G)}(x) \cap D' \neq \emptyset$

If $d=2$ and at least one of the components in $H-x$ is isomorphic to $G_{10}$ the set $D'$ can be chosen such that $N(x) \cap D' \neq \emptyset$.

Let $U_k$ denote the vertices from $T-H$ at distance $k$ from $x$. It follows that $D_i := D'_i \cup (U_{d+1} \cap V_i)$ dominates $V_i$.

Further define $I := 1$ if there exists a vertex $v$ in a component $C$ of $T - H$ such that $d(v, x) = \max_{u \in V(C)} d(u, x) \leq d$ and $d(v, D') > d$ and otherwise let $I := 0$. Let $U_1' = U_1 \cap N_{(d,G)}[U_{d+1}]$ and define $D$ by $D := D' \cup U_1'$ if $I = 0$ and $D := D' \cup U_{1'} \cup \{x\}$ if $I = 1$. Now $D$ is defined such that it distance $d$ dominates $T$. Therefore

$$f_d(3, T) \leq |D| + \sum_{i=1}^{3} |D_i'| = I + \gamma_d(H - x) + |U_{d+1}| + \sum_{i=1}^{3} |D_i'|.$$

First assume that $d = 2$. By the induction hypothesis it follows that $f_d(3, T) < n$ and thus the theorem easily follows if $n \leq 30$ since $f_d(3, T) \leq n - 1$ and $n - 1 < \frac{30}{31} n$ for $n \leq 30$. Thus it can be assumed that $n \geq 31$ in this case.

If $deg_{E_0}(x) \leq 2$ we obtain that

$$\begin{aligned}
f_d(3, T) &\leq 3 - deg_{E_0}(x) + g_d(3, H - x) + \gamma_d(H - x) + 2|U_{d+1}| + I \\
&\leq 3 + f_d(3, H - x) + 2|U_{d+1}| \\
&\leq 3 + f_d(3, H - x) + \frac{2}{d+1}(n - |V(H)|) \\
&\leq 3 + |V(H - x)| + \frac{2}{d+1} 9 + \frac{2}{d+1}(n - |V(H)| - 9) \\
&\leq \frac{29}{30} 30 + \frac{2}{3}(n - 30) \\
&< \frac{30}{31} n.
\end{aligned}$$

If $deg_{E_0}(x) \geq 3$ the following is obtained :

$$\begin{aligned}
f_d(3, T) &\leq 3 - deg_{E_0}(x) + f_d(3, H - x) + I + 2|U_{d+1}| \\
&\leq \frac{30}{31}(|V(H)| + I) + \frac{2}{d+1}(n - (|V(H)| + I)) \\
&\leq \frac{30}{31} n.
\end{aligned}$$

Further, equality holds if and only if $n = |V(H)| = 31$ and in this case we have that $T \cong T_2$.

Assume that $d \geq 3$. If $deg_{E_0}(x) \leq 2$ and $n \geq 6d+13$ then

$$\begin{aligned}
f_d(3,T) &\leq 3 - deg_{E_0}(x) + g_d(3, H-x) + \gamma_d(H-x) + \frac{2}{d+1}(n - |V(H)|) \\
&\leq 2 + f_d(3, H-x) + \frac{2}{d+1}(n - |V(H)|) \\
&\leq 2 + \frac{24}{6d+12}(|V(H)| - 1) + \frac{2}{d+1}(2d+4) \\
&\quad + \frac{2}{d+1}(n - |V(H)| - (2d+4)) \\
&\leq 2 + \frac{24}{6d+12}(4d+8) + \frac{2}{d+1}(2d+4) + \frac{2}{d+1}(n - 6d - 13) \\
&\leq 23 + \frac{2}{d+1}(n - 6d - 13) \\
&< \frac{24}{6d+13}(6d+13) + \frac{24}{6d+13}(n - (6d+13)) = \frac{24}{6d+13}n.
\end{aligned}$$

If $deg_{E_0}(x) \geq 3$ then

$$\begin{aligned}
f_d(3,T) &\leq 3 - deg_{E_0}(x) + g_d(3, H-x) + \gamma_d(H-x) + \frac{2}{d+1}(n - |V(H)|) \\
&\leq \frac{24}{6d+13}|V(H)| + \frac{2}{d+1}(n - |V(H)|) \\
&\leq \frac{24}{6d+13}n.
\end{aligned}$$

Here equality holds if and only if $deg_{E_0}(x) = 3$ and $n = |V(H)| = 6d+13$. Thus it follows that equality holds if and only if $T \cong T_d$.

Assume that $n \leq 6d+12$ and $deg_{E_0}(x) \leq 2$. By the choice of $x$ it follows that the graph $G' = G - V(H-x)$ has $diam(G') \leq 2d+2$. Assume that $deg_{E_0}(x) = 1$ then it follows from the assumptions for the set $E_0$ that $|V(G')| \geq d+3$.

If $diam(G') \leq 2d$ then $f_d(3, G') = 4 \leq \frac{4}{d+2}(|V(G')| - 1)$ and $f_d(3, G) \leq f_d(3, G') + f_d(3, G - V(G')) \leq \frac{4}{d+2}(|V(G)| - 1) < \frac{24}{6d+13}|V(G)|$. Assume that $diam(G') = 2d+1$ and $|V(G')| \geq 2d+3$. It follows that $f_d(3, G') = 6 \leq \frac{4}{d+2}(|V(G')| - 1)$ and from this we obtain that $f_d(3, G) < \frac{24}{6d+13}|V(G)|$. If this is not the case then $G' \cong P_{2d+2}$ and $x$ is a central vertex in $G'$ and it follows from Observation 2 that $f_d(3, G) \leq f_d(3, H-x) + f_d(3, G') - 1 \leq \frac{4}{d+2}|V(G)| - 1 \leq \frac{4}{d+2}(|V(G)| - 1) < \frac{24}{6d+13}|V(G)|$. Thus it can be assumed that $diam(G') = 2d+2$. In this case it was proven in Lemma 1 that for each partition $(V_1', V_2', V_3')$ of $G'$ where $f_d(3, G') = \gamma_d(G') + \sum_{i=1}^{3} \gamma_d(G; V_i')$ there exists sets $X_i$ such that $x \in X_i$, $\sum_{i=1}^{3} |X_i| \leq \frac{24}{6d+13}|V(G')|$ and $V_i' \subseteq N_d(X_i)$. From this and Observation 2 it follows that $f_d(3, G) \leq f_d(3, H-x) + \sum_{i=1}^{3} |X_i| - 1 < \frac{24}{6d+13}|V(G)|$.

Thus it can be assumed that $deg_{E_0}(x) = 2$ and $|V(G)| \leq 6d+13$. If $|V(G')| = 1$ then let $C$ be a component from $G-x$ and let $C'$ be the induced subgraph of $G$ containing $V(C) \cup \{x\}$. It follows by the induction hypothesis that $f_d(3, C') = f_d(3, C)$ and thus $f_d(3, G) \leq f_d(G-x) \leq \frac{4}{d+2}(|V(G)|-1) < \frac{24}{6d+13}|V(G)|$. Thus it can be assumed that $|V(G')| \geq 2$.

First it is assumed that the two components of $H-x$ are not both isomorphic to $P_{d+2}$. Thus it can be assumed that $D_i \cap N[x] \neq \emptyset$ for each $i \in \{1,2,3\}$. It follows that $f_d(3,G) \leq \frac{4}{d+2}|V(H-x)| + \frac{3}{d+1}|V(G-H)| < \frac{24}{6d+13}|V(G)|$. Thus the theorem has been proved in this sub case and it can be assumed that both components of $H-x$ are isomorphic to $P_{d+2}$. If $|V(G')| < d-1$ it can easily be seen that $f_d(3,G) = f_d(3, H-x)$ and the theorem easily follows. If this is not the case we obtain that

$$\begin{aligned}
f_d(3,T) &\leq f_d(H-x) + 1 + \frac{2}{d+1}(n - |V(H)|) \\
&= 9 + \frac{2}{d+1}(d-1) + \frac{2}{d+1}(n - (2d+5) - (d-1)) \\
&< \frac{24}{6d+13}(3d+4) + \frac{2}{d+1}(n - (3d+4)) \\
&\leq \frac{24}{6d+13}n.
\end{aligned}$$

Thus the statement about $f_d(3,T)$ has been proved in this sub case.

We shall now consider $g_2(3,T)$ in case 3. If $T \in Q'_d$ then the theorem easily follows by verification. Otherwise It follows that $f_d(3,T) \leq n-1$ and since $g_d(3,T) \leq \frac{3}{4}f_d(3,T)$ we obtain that $g_d(3,T) \leq \lfloor \frac{3}{4}(n-1) \rfloor$. From this inequality it follows that $g_d(3,T) < \frac{18}{25}n$ if $n < 25$. Thus it can be assumed that $n \geq 25$. Since $D_i$ is a distance $d$ dominating set for $V_i$ we have that

$$g_2(3,T) \leq \sum_{i=1}^{3} |D_i| \leq 3 - deg_{E_0}(x) + g_2(3, H-x) + \frac{1}{d+1}(n - |V(H)|).$$

If $deg_{E_0}(x) \leq 2$ then $|V(H)| \leq 2(2d+4)+1 = 17$ and it follows that

$$\begin{aligned}
g_2(3,T) &\leq 3 - deg_{E_0}(x) + g_2(3, H-x) + \frac{1}{d+1}(n - |V(H)|) \\
&\leq 2 + \frac{3}{4}17 + \frac{1}{d+1}8 + \frac{1}{d+1}(n - 17 - 8) \\
&< \frac{18}{25}25 + \frac{18}{25}(n - 17 - 8) = \frac{18}{25}n.
\end{aligned}$$

If $deg_{E_0}(x) \geq 3$ then

$$\begin{aligned} g_2(3,T) &\leq 3 - deg_{E_0}(x) + g_2(3, H - x) + \frac{1}{d+1}(n - |V(H)|) \\ &\leq \frac{18}{25}|V(H)| + \frac{18}{25}(n - |V(H)|) = \frac{18}{25}n, \end{aligned}$$

and if equality holds then $deg_{E_0}(x) = 3$ and $n = |V(H)| = 6d + 13 = 25$. From the observations done so for we obtain that $H \cong T_2$ if equality holds.

So far we have not considered the statement about $g_d(3,T)$, for $d \geq 3$, in any cases. The statement is easily verified if $T \in Q'_d \cup \mathcal{F}'_d$ and otherwise the statement proved for $f_d(3,T)$ can be used to obtain $g_d(3,T) \leq \frac{3}{4} f_d(3,T) < \frac{18}{6d+13} n$. □

# 4 Many partition classes

**Lemma 3.** Let $k$ and $d$ be positive integers and let $T$ be a tree with $n$ vertices. Then $g_d(k,T) = n$ if and only if $|N_{(d,T)}[v]| \leq k$ for each vertex $v \in V(T)$.

*Proof.* For $v \in V(T)$ the subgraph of $T$ induced by $N_{(d,G)}[v]$ is denoted $T_v$. Let $T'$ be the graph where $V(T') = V(T)$ and $E(T') = \{uv | V(T_u) \cap V(T_v) \neq \emptyset\}$. Since $T$ is a tree $T'$ is a chordal graph. Clearly the chromatic number of $T'$ equals the minimum number of $2d$-independent sets into which $V(T)$ can be partitioned. Since $T'$ is a chordal graph it is perfect and we have that its chromatic number equals its clique number, $\chi(T') = \omega(T')$.

Now let $\{v_1, \ldots, v_a\}$ be a clique in $T'$, i.e., a subset of $V(T)$ such that $V(T_{v_i}) \cap V(T_{v_j}) \neq \emptyset$ for all $i, j, 1 \leq i \leq j \leq \alpha$, it then follows that there must exist a vertex $v$ such that $v \in V(T_{v_i})$ for $i \in \{1, \ldots, a\}$. Let namely $S$ denote the subtree of $T$ spanned by $\{v_1, \ldots, v_a\}$. Let $v$ be a central vertex for a longest path in $S$, then $v$ has distance $\leq d$ to all $v_i$, $1 \leq i \leq a$, and $v \in \bigcap_{i=1}^{a} T_{v_i}$. From this observation we have that $g_d(k,T) = n$ if and only if $|N_d[x]| \leq k$ for each $x$ in $V(T)$. □

**Lemma 4.** Let $H$ be a connected graph with at least $2d + 1$ vertices and let $G = H \circ P_d$ be the $P_d$-corona graph of $H$. Then there exist vertices $v_H \in V(H)$ and $v_G \in V(G)$ such that

$$|N_{(d,H)}[v_H]| \geq 2d + 1 \qquad \text{and} \qquad |N_{(d,G)}[v_G]| \geq (d+1)^2.$$

*Proof.* Let $H$ be a connected graph such that $|V(H)| \geq 2d+1$ and let $G = H \circ P_d$, i.e., $G$ is obtained by joining each vertex of $H$ to an end of its own copy of a

$P_d$, and $G$ has order $|H|(d+1)$. Let $v_H$ be a central vertex (a vertex with minimum eccentricity) in $H$. If $\gamma_d(H) = 1$ then $N_{(d,H)}[v_H] = V(H)$ and we obtain that $|N_{(d,H)}[v_H]| \geq 2d+1$. If $\gamma_d(H) \neq 1$ then there must be a path $P : v_1, \ldots, v_d, v_H, v_{d+2}, \ldots, v_{2d+1}$ in $H$ and since $V(P) \subseteq N_{(d,H)}[v_H]$ it can be concluded that $|N_{(d,H)}[v_H]| \geq 2d+1$.

Let $a_i$ denote the number of vertices in $H$ at distance $i$ from $v_H$ then

$$|N_{(d,G)}[v_H]| = (d+1) + \sum_{i=1}^{d}(d+1-i)a_i.$$

If $a_i \geq 2$ for each $i \in \{1, \ldots, d\}$ then

$$|N_{(d,G)}[v_H]| \geq (d+1) + \sum_{i=1}^{d} 2(d+1-i) = (d+1)^2.$$

If $a_k \leq 1$ for an index $k \in \{1, \ldots, d\}$ then $a_i = 0$ for $i > k$ since $v_H$ is a central vertex. Since $\sum_{i=1}^{d} a_i \geq 2d$ it follows that

$$\begin{aligned}|N_{(d,G)}[v_H]| &= (d+1) + \sum_{i=1}^{d}(d+1-i)a_i = (d+1) + \sum_{i=1}^{k}(d+1-i)a_i \\ &\geq (d+1) + \sum_{i=1}^{k-1} 2(d+1-i) + (2d-2(k-1))(d+1-k) \geq (d+1)^2.\end{aligned}$$

$\square$

From Theorem 1, Observation 1, Lemma 3 and Lemma 4 the following result is easily obtained.

**Corollary 1.** Let $d \geq 1$ be an integer and let $T$ be a tree with $n$ vertices then

- $f_d((d+1)^2, P_{\frac{n}{d+1}} \circ P_d) = \frac{d+2}{d+1}n$ if $(d+1) \mid n$.
- $g_d(2d+1, P_n) = n$ for each $n \geq 1$.
- $g_d(d^2+2d, T) < n$ if $T$ is a $P_d$-corona graph and $|V(T)| > 2d(d+1)$.
- $f_d(d^2+2d, T) < \frac{d+2}{d+1}n$ if $|V(T)| > 2d(d+1)$.
- $f_d(d^2+2d, P_{2d} \circ P_d) = \frac{d+2}{d+1}n$.
- $g_d(2d, T) < n$ if $|V(T)| \geq 2d+1$.

**Lemma 5.** If $T$ is a tree, $X \subseteq V(T)$ and $d \geq 1$ is an integer then there exists a set $X' \subseteq X$ such that $X'$ is a $2d$-independent in $T$ and $|X'| = \gamma_d(T;X)$.

*Proof.* This result is proven on induction on $\gamma_d(X)$. If $\gamma_d(T;X) \leq 1$ the theorem is trivially true. Assume that $\gamma_d(T;X) \geq 2$. Let $P = x_1, v_1, \ldots, v_a, x_2$ be a path in $T$ of maximum length when $\{x_1, x_2\} \subseteq X$. Since $\gamma_d(T; X \backslash N_{(d,T)}[v_d]) \geq \gamma_d(T;X) - 1$ the induction hypothesis gives that there exists a $2d$-independent set $X' \subseteq X \backslash N_{(d,T)}[v_d]$ in $T - N_{(d,T)}[v_d]$ with cardinality $\gamma_d(X) - 1$. By the choice of $P$ it must hold that $X' \cup \{x_1\}$ is a $2d$-independent set in $T$ and the result follows. □

**Theorem 3.** Let $d \geq 1$ be an integer and let $T$ be a tree with $n > 2d^2 + 2d$ vertices then

$$f_d(d^2 + 2d, T) < \frac{d+2}{d+1} n - \frac{n}{2(d+1)^5}.$$

*Proof.* Let $T$ be a tree with $n > 2d^2 + 2d$ vertices. If $diam(T) \leq 2d$ then $f_d(d^2 + 2d, T) = d^2 + 2d + 1 \leq \frac{d+2}{d+1} n - \frac{n}{(d+1)^2}$. Thus it can be assumed that $diam(T) \geq 2d + 1$ and further we assume that the theorem does not hold for $T$.

Let $k = 2(d+1)^5$ then

$$f_d(d^2 + 2d, T) \geq \frac{d+2}{d+1} n - \frac{n}{k}.$$

From Theorem 1 follows that there must exist a constant $a \in [0,1]$ such that $\gamma_d(T) \geq \frac{n}{d+1} - a\frac{n}{k}$ and $g_d(d^2 + 2d, T) \geq n - (1-a)\frac{n}{k}$.

Let $X_1, \ldots, X_{d^2+2d}$ be a partition of $V(T)$ such that

$$g_d(d^2 + 2d, T) = \sum_{i=1}^{d^2+2d} \gamma_d(T; X_i).$$

From Lemma 5 follows that for each $i \in \{1, \ldots, d^2 + 2d\}$ there exists a set $X_i' \subseteq X_i$ such that $\gamma_d(T; X_i) = |X_i'|$ and $X_i'$ is a $2d$-independent set in $T$. Hence

$$\sum_{i=1}^{d^2+2d} |X_i'| = g_d(d^2 + 2d, T) \geq n - (1-a)\frac{n}{k}.$$

Let $H := \{v \in V(T) | deg(v) \geq (d+1)^2\}$ then we shall prove below that

$$\sum_{v \in H} (deg(v) - (d+1)^2) \leq (1-a)\frac{n}{k}.$$

Root the tree $T$ at a vertex $x \in V(T)$ and let $c(v)$ denote the children of a vertex $v$ in the rooted tree $T$. Let $v \in H$ then $c(v)$ contains at least $deg(v) - 1$ vertices and since each set $X'_i$ can contain at most one vertex from $c(v)$ we have that

$$|c(v) \backslash (\bigcup_{i=1}^{d^2+2d} X'_i)| \geq deg(v) - 1 - (d^2 + 2d) = deg(v) - (d+1)^2.$$

Since $c(v_1) \cap c(v_2) = \emptyset$ if $v_1 \neq v_2$ and $\sum_{i=1}^{d^2+2d} |X'_i| \geq n - (1-a)\frac{n}{k}$ it must hold that

$$(1-a)\frac{n}{k} \geq \sum_{v \in H} |c(v) \backslash (\bigcup_{i=1}^{d^2+2d} X'_i)| \geq \sum_{v \in H} \left(deg(v) - (d+1)^2\right).$$

This proves the inequality.

Let $S$ be a maximum $2d$-independent set in $T$. From Lemma 5 it follows that $|S| = \gamma_d(T) \geq \frac{n}{d+1} - a\frac{n}{k}$. Consider for each $s \in S$ the tree $T_s := T\left[N_d[s]\right]$ spanned in $T$ by $N_d[s]$. Since $S$ is $2d$-independent no vertex from $T$ is in more than one of these trees. From the assumption that $diam(T) \geq 2d+1$ it follows that there must be a path $P_s = s, v_2, \ldots, v_{d+1}$ in $T_s$. It follows that $\sum_{s \in S} |V(P_s)| = |S|(d+1) \geq n - (d+1)a\frac{n}{k}$. Let $F' := \bigcup_{s \in S} V(P_s)$ and let $A = V(T) - V(F')$. Let $B$ be all vertices from those paths $P_s$ for which $P_s \not\subseteq \bigcup_{i=1}^{d^2+2d} X'_i$ and let $B'$ be the set of end vertices not in $S$ from these paths.

In the following we examine the number of vertices and components in the induced subgraph of $T$ with vertex set $F := F' - B$. We observe that both $T[F]$ and $T[F']$ are $P_d$-corona graphs, as they are obtained by adding edges between some end vertices, not in $S$, of the $P_s$-paths. Each $P_s \in B$ contains a vertex of $V(T) \setminus \bigcup_{i=1}^{d^2+2d} X'_i$, so from $|V(T) \setminus \bigcup_{i=1}^{d^2+2d} X'_i| \leq (1-a)\frac{n}{k}$ and $|V(P_s)| = d+1$ we get $|B| \leq (d+1)(1-a)\frac{n}{k}$.

By the assumptions we have that

$$\begin{aligned}|F| = |F'| - |B| &\geq |F'| - (d+1)(1-a)\frac{n}{k} \\ &\geq (n - (d+1)a\frac{n}{k}) - (d+1)(1-a)\frac{n}{k} = n - (d+1)\frac{n}{k}.\end{aligned}$$

From the following calculations we obtain an upper bound on the number of components in $F$.

$$\omega(F) \leq \sum_{v \in A \cup B'} deg(v) = \sum_{v \in (A \cup B') \setminus H} deg(v) + \sum_{v \in (A \cup B') \cap H} deg(v)$$

$$\leq (d+1)^2 |(A \cup B') \setminus H| + (d+1)^2 |(A \cup B') \cap H| + \sum_{v \in H} (deg(v) - (d+1)^2)$$

we apply $|A| \leq (d+1)a\dfrac{n}{k}$ and $|B'| \leq \dfrac{|B|}{d+1} \leq (1-a)\dfrac{n}{k}$ to obtain

$$\omega(F) \leq (d+1)^2 |A \cup B'| + (1-a)\frac{n}{k} \leq \frac{n}{k}((1-a) + (1-a)(d+1)^2 + a(d+1)^3).$$

Since $V(F) \subseteq \bigcup_{i=1}^{d^2+2d} X_i'$ we have that $F$ is a $P_d$-corona graph which satisfies $g_d(d^2+2d, F) = |V(F)|$. By Corollary 1 this can only hold if each component of $F$ has at most $2d(d+1)$ vertices and we obtain that

$$\frac{n - (d+1)\frac{n}{k}}{\frac{n}{k}((1-a) + (1-a)(d+1)^2 + a(d+1)^3)} \leq 2d(d+1).$$

From this equation we easily obtain the contradiction that $k < 2(d+1)^5$. $\square$

**Theorem 4.** Let $G$ be a tree with $n \geq d + \frac{k+1}{2}$ vertices then

$$g_d(k, G) \leq n\frac{2k}{2d+k+1}.$$

*Proof.* The theorem easily follows when $k > 2d+1$ since $g_d(k,G) \leq n < n\frac{2k}{2d+k+1}$ in this case. For $k \leq 2d+1$ the theorem is proven by induction on $n$. The theorem easily follows if $n = \lceil d + \frac{k+1}{2} \rceil$ since $\gamma_d(G) = 1$ in this case. If the graph $G$ has an edge $e$ such that both components, $G_1$ and $G_2$, of $G - e$ have at least $d + \frac{k+1}{2}$ vertices the induction hypothesis can be used on both components to obtain inequality.

Thus it can be assumed that the removal of each edge in $G$ gives a component with less than $d + \frac{k+1}{2}$ vertices. Let $e$ be an edge in $G$ such that one of the components, $G_1$, in $G - e$ has a maximum number of vertices when the other $G_2$ must contain $d + \frac{k+1}{2}$ vertices. Let $u$ be the vertex from $G_2$ incident to $e$. By the choice of $e$ it follows that the maximum distance from $u$ to a vertex in $G_2$ is at most $d + \frac{k-1}{2}$. By using the induction hypothesis on $G_2$ it can be observed that to each partition $V_1, \ldots, V_k$ of $G_2$ there are related dominating sets $D_1, \ldots, D_k$ such that $g_d(k, G) \leq \sum_{i=0} |D_i| \leq |V(G_2)|\frac{2k}{2d+k+1}$ and each set $D_i$ contains a vertex from $N_{\lfloor \frac{k-1}{2} \rfloor}[u]$ and if $A := \bigcap_{i=1}^{k} N_{(d,G)}[D_i] \cap G_1$ then $|A| \geq \lceil d - \frac{k-1}{2} \rceil$ and $g_d(k, G) \leq \sum_{i=1}^{k} |D_i| + |V(G_1)| - |A|$ and calculations gives :

$$\frac{|V(G_1)| - |A|}{|V(G_1)|} \leq \frac{|V(G_1)| - \lceil d - \frac{k-1}{2} \rceil}{|V(G_1)|} < \frac{d + \frac{k+1}{2} - (d - \frac{k-1}{2})}{d + \frac{k+1}{2}} = \frac{2k}{2d+k+1}.$$

Thus it follows that $g_d(k,G) \leq g_d(k,G_2) + |V(G_1)| - |A| \leq |V(G_2)|\frac{2k}{2d+k+1} + |V(G_1)|\frac{2k}{2d+k+1} = n\frac{2k}{2d+k+1}$. $\square$

From Theorem 4 follows

**Corollary 2.** A graph $G$ with $n \geq 2d+1$ vertices satisfy that
$g_d(2d,G) \leq n - \frac{n}{4d+1}$.

In [20] it has been proven that this bound is optimal when $d = 1$.

## References


[1] N. Alon and J.H. Spencer, *The Probabilistic Method*, John Wiley and Sons, Inc., 1992.

[2] V.I. Arnautov, Estimation of the exterior stability number of a graph by means of the minimal degree of the vertices, (in Russian) *Prikl. Mat. i Programmiravanie Vyp. 11*(1974), 3-8, 126.

[3] Y. Caro and Y. Roditty, On the vertex-independence number and star decomposition of graphs, *Ars Combin.* **20** (1985), 167-180.

[4] Y. Caro and Y. Roditty, A note on the k-domination number of a graph *Internat. J. Math. Sci.* **13** (1990), 205-206.

[5] G. Chartrand and L. Lesniak, *Graphs and Digraphs: Third Edition*, Chapman & Hall, London, 1996.

[6] J.F. Fink, M.S. Jacobson, L.F. Kinch and J. Roberts, On graphs having domination number half their order. *Period. Math. Hungar.* **16** (1985), 287-293.

[7] C-M. K. Fu and P.D. Vestergaard, "Distance domination in partitioned graphs", *Congressus Numerantium*, (2006) pp. 1-5, manuscript.

[8] B.L. Hartnell and P.D. Vestergaard, Partitions and dominations in a graph. *J. Combin. Math. Combin. Comput.* **46** (2003), 113–128.

[9] T.W. Haynes, S.T. Hedetniemi, and P.J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, New York, 1998.

[10] T.W. Haynes, S.T. Hedetniemi, and P.J. Slater (eds.), *Domination in Graphs: Advanced Topics*, Marcel Dekker, New York, 1998.

[11] M.A. Henning, O.R. Oellermann and H.C. Swart, Bounds on distance domination parameters. *J. Combin. Inform. System Sci.* **16** (1991), 11-18.

[12] M.A. Henning and P.D. Vestergaard,"Domination in partitioned graphs with minimum degree two"', pp. 1-29. To appear in *Discrete Mathematics.*

[13] W. McCuaig and B. Shepherd, Domination in graphs with minimum degree two. *J. Graph Theory* **13** (1989), 749–762.

[14] A. Meir and J.W. Moon, Relations beween packing and covering numbers of a tree. *Pacific Journal of Mathematics* **61**, (1975), 225-233.

[15] O. Ore, Theory of Graphs, Amer. Math. Soc. Colloq. Publ., **38**. Amer. Math. Soc., Providence, RI, 1962.

[16] C. Payan, Sur le nombre d'absorption d'un graph simple, *Cahiers Centre Étude Rech. Opér.* **2.3.4**, (1975), 171.

[17] C. Payan and N. H. Xuong, Domination-balanced graphs, *Journal of Graph Theory*, **6** (1982), 23-32.

[18] B. Reed, Paths, stars and the number three. *Combin. Probab. Comput.* **5 (3)** (1996), 277-295.

[19] S.M. Seager, Partition dominations of graphs of minimum degree 2. *Congress. Numer.* **132** (1998), 85–91.

[20] Z. Tuza and P.D. Vestergaard, Domination in partitioned graph. *Discussiones Mathematicae Graph Theory.* **22**(1) (2002), 199–210.

# Total Domination (Without Isolated Vertices) in Partitioned Graphs

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**Abstract**

Let $G = (V, E)$ be a graph with no isolated vertex. For sets $S, V' \subseteq V(G)$ the set $S$ is said to total dominate $V'$ if $V' \subseteq N[S]$ and each vertex from $S$ is adjacent to another vertex from $S$. The minimum cardinality of a set that total dominates $V'$ is denoted $\gamma_t(V')$. For a graph $G$ and a partition of $V(G)$ into $k$ sets $V_1, \ldots, V_k$ such that $\gamma_t(V_1) + \ldots + \gamma_t(V_k)$ is maximum we define $f_t(k, G) := \gamma_t(V_1) + \ldots + \gamma_t(V_k)$.

In this paper tight upper bounds for $f_t(2, G)$ is given when $G$ is a tree and when $G$ is a graph with minimum degree at least two. Further we characterize the extremal graphs for these bounds.

**Keywords:** total domination, decomposition/composition, total well dominated
**AMS subject classification: 05C69**

## 1 Notation

For a graph $G = (V(G), E(G))$ and two set of vertices $V', S \subseteq V(G)$ then $S$ is said to dominate $V'$ if $V' \subseteq N[S]$. If we further have that $s \in N(S)$ for each vertex $s \in S$ then we say that $S$ is a total dominating set for $V'$. If $S$ dominates $V(G)$ then we say that S is a dominating set for $G$ and if $S$ is a total dominating set for $V(G)$ then we say that S is a total dominating set for $G$. The minimum cardinality of a dominating set for $V' \subseteq V(G)$ is denoted by $\gamma(V')$ and the minimum cardinality of a total dominating set for $V' \subseteq V(G)$ is denoted by $\gamma_t(V')$. Further we define $\gamma(G) := \gamma(V(G))$ and $\gamma_t(G) := \gamma_t(V(G))$. For a graph $G$ and a integer $k \geq 2$ we define $f(k, G)$ and $f_t(k, G)$ by

$f(k,G) := \max\{\gamma(V_1)+\cdots+\gamma(V_k) | V_1 \cup \cdots \cup V_k = V(G), V_i \cap V_j = \emptyset \text{ for } i \neq j\}$.
$f_t(k,G) := \max\{\gamma_t(V_1)+\cdots+\gamma_t(V_k) | V_1 \cup \cdots \cup V_k = V(G), V_i \cap V_j = \emptyset \text{ for } i \neq j\}$.
For $k = 2$ we denote $f(k,G)$ by $f(G)$ and $f_t(k,G)$ by $f_t(G)$.

# 2 Complexity-results for $f(k,G)$ and $f_t(k,G)$

In this section we show that it is NP-hard to determine $f(k,G)$ and $f_t(k,G)$ by using the well known result, that it is NP-complete to determine $\gamma(G)$.

**PROBLEM :** DOMINATING SET (DOM)
**INSTANCE :** A graph $G$ and a positive integer $j$.
**QUESTION :** Is $\gamma(G) \leq j$.

In [5] the following result was proven.

**Theorem 1.**
The problem DOM is NP-complete.

In the following we prove that for each integer $k \geq 2$ we have that the problem $k$-PARTITONED DOMINATION is NP-complete.

**PROBLEM :** $k$-PARTITONED DOMINATION ($k$-PART)
**INSTANCE :** A graph $G$ and a positive integer $j$.
**QUESTION :** Is $f(k,G) \leq j$.

**Theorem 2.**
For each integer $k \geq 2$ the problem $k$-PART is NP-hard.

*Proof.* To prove the theorem we use that DOM is NP-complete. Given a graph $G$ we construct the graph $G'$ from $G$ by adding $k$ vertices $v'_1, \ldots, v'_k$ for each vertex $v \in V(G)$ and edges from $v'_1, \ldots, v'_k$ to each vertex from $N_G[v]$. Let $V_1, \ldots, V_k$ be a partition of $V(G)$ such that $V_i$ contains $v'_i$ for each vertex $v \in V(G)$. Since $N[v'_i] \subseteq N[v]$ there exists $\gamma(V_i)$-sets for $G'$ that only consists of vertices from $G$. By the construction of $G'$ we have that these $\gamma(V_i)$-sets for $G'$ must contain a vertex from $N[v]$ for each vertex $v \in V(G)$ since $v'_i \in V_i$. Thus $f(k,G') \geq k\gamma(G)$. Let $V_1, \ldots, V_k$ be a partition of $V(G')$ such that $f(k,G') = \gamma(V_1)+\cdots+\gamma(V_k)$. Since a dominating set $S$ in $G$ dominates $G'$ we have that

$$f(k,G') = \gamma(V_1)+\cdots+\gamma(V_k) \leq k\gamma(G') \leq k\gamma(G).$$

From these observations we have that $f(k,G) = k\gamma(G)$, and we must have that $k$-PART is NP-hard since DOM is NP-compelete. $\square$

**PROBLEM :** TOTAL $k$-PARTITONED DOMINATION ($k$-PARTTOT)
**INSTANCE :** A graph $G$ and a positive integer $j$.
**QUESTION :** Is $f_t(k, G) \leq j$.

**Theorem 3.**
For each integer $k \geq 2$ the problem $k$-PARTTOT is NP-hard.

*Proof.* To prove the theorem we use that $k$-PART is NP-hard. Given a graph $G$ we construct the graph $G' = (V(G') = V(G) \cup \{v' | v \in V(G)\}, E(G') = \{vx' | v \in V(G) \text{ and } x \in N[v]\})$. From $G'$ we construct the graph $G''$ by adding a star $S \cong K_{1,k+1}$ and edges from one of the end vertices from $S$ to each vertex in $V(G)$. It can now be proven that $f_t(k, G'') = 2k + f(k, G)$. Let $V_1, \ldots, V_k$ be a partition of $V(G'')$ such that each set $V_i$ contains a end vertex from the star $S$. It is easily seen that $\sum_{i=1}^{k} \gamma_t(V_i) = 2k + \gamma(V_i \cap V(G')) \leq 2k + f(k, G)$ where equality will hold for one of these partitions. We obtain that $f_t(k, G'') = 2k + f(k, G)$. Since $k$-PART is NP-hard we have that $k$-PARTTOT is NP-hard. $\square$

## 3 Bounds for $f_t(G)$

In this section tight upper bounds for $f_t(2, G)$ is given when $G$ is a tree and when $G$ is a graph with minimum degree at least two.

In [3] the following theorem was proven.

**Theorem 4.**
Let $G$ be a connected graph of order $n \geq 3$, then $\gamma_t(G) \leq \frac{2}{3}n$ and equality holds if and only if $G \cong C_3, G \cong C_6$ or $G$ is the 2-corona graph of a connected graph.

First we prove a result for $f_t(T)$, when $T$ is a tree with diameter at most six. In the following theorem we denote by $G_1$ the graph obtained from $K_{1,3}$ by subdividing one of the edges three times. By attaching a graph $H$ with an end vertex $u$ to a vertex $v$ in a graph $G$, we shall mean the graph obtained by adding the edge $uv$ to $G \cup H$ and afterwards contracting $uv$.

**Theorem 5.**
Let $T$ be a tree of order $n \geq 4$ such that at least one component of $T - e$ has fewer than four vertices for each edge $e \in E(T)$. Then $f_t(T) \leq n + 1$ and

i) $f_t(T) = n + 1$ if and only if $n = 3k + 1$ for an integer $k \geq 1$ and $T$ can be obtained from the star $K_{1,\frac{n-1}{3}}$ by subdividing each edge two times.

ii) $f_t(T) = n$ if and only if $T$ is one of the graphs $K_{1,3}$, $P_4$, $P_5$, $P_6$, $G_1$ or $T$ can be constructed from one of the graphs from i) with at least seven vertices by attaching a $P_2, P_3$ or $K_{1,3}$ to the center.

*Proof.* By the assumptions for $T$ we have that $2 \leq d \leq 6$ when $d$ denotes the diameter of $T$. Let $P_{d+1} : v_1, v_2, \ldots, v_{d+1}$ be a diametrical path in $T$.

If $d = 2$ then $T$ is a star and $f_t(T) = 4 \leq n$. If $d = 3$ then $\{v_2, v_3\}$ is a total dominating set for $T$ and we may conclude that $f_t(T) \leq 4 \leq n$. Thus we may assume that $d \geq 4$.

For $d = 4$ we consider a partition $(V_1, V_2)$ of $V(G)$ such that $\gamma_t(V_1) + \gamma_t(V_2) = f_t(T)$. If $v_2$ is adjacent to more than one end vertex then it can be assumed that not all of these vertices are in $V_1$ or $V_2$.

Thus $S_i = \{v_2, v_3\} \cup \{v \in N(v_3) \backslash \{v_2\} | v \text{ is adjacent to an end vertex } u \in V_i\}$ is a total dominating set for $V_i$ and $|S_1| + |S_2| < n$. Thus it can be assumed that $T$ can be obtained form a star $K_{1,s}$ by subdividing at least two edges two times and the rest of the edges one time. If one of the sets $V_1$ or $V_2$ from the partition of $V(G)$ contain all vertices at distance two from $v_3$ it is total dominated by $N[v_3] \backslash L$ if $L$ is the leaves in $T$ and the other set from the partition is total dominated by $\{v_2, v_3\}$. Thus in this case $f_t(T) \leq n$ and equality holds if and only if $T \cong P_5$. If this is not the case then it can be assumed that $v_1 \in V_1$ and $v_5 \in V_2$. Define $S_i = \{v_3\} \cup \{v \in N(v_3) | v \text{ is adjacent to an end vertex } u \in V_i\}$ then $S_i$ is a total dominating set for $V_i$ and $|S_1| + |S_2| < n$. Thus it can be assumed that $d \in \{5, 6\}$.

Assume that $d = 5$ and let $(V_1, V_2)$ be a partition of $V(T)$. By the assumptions for $T$ we can assume that $deg(v_4) = deg(v_5) = 2$. Let $T'$ be the induced subgraph of $T$ containing $V(P_{d+1})$ and all vertices adjacent to $v_2$. It can be observed that there exists total dominating sets $S'_i$ for $V_i \cap V(T')$ containing $v_3$ such that $|S'_1| + |S'_2| \leq |V(T')|$. Since

$$S_i := S'_i \cup \{v \in N(v_3) \backslash \{v_2, v_4\} | v \text{ is adjacent to an end vertex } u \in V_i\}$$

is a total dominating set for $V_i$ it can be observed that $f_t(T) \leq |S_1| + |S_2| \leq n$ where we must have that $deg(v_3) = 2$ if equality holds. From this observation we have that $T$ can be obtained from $P_{d+1}$ by attaching paths of length one to $v_2$ if $f_t(T) = n$. If more than two paths are attached to $v_2$ it easily follows that $f_t(T) < n$. Thus the theorem holds when $d = 5$.

If $d = 6$ then by the assumptions for $T$ we must have that $T$ can be obtained from $P_{d+1}$ by attaching paths of length three and stars to $v_4$. It can be observed that if $T'$ is the maximal subgraph of $G$ that can be obtained from $P_{d+1}$ by attaching paths of length three then $f_t(T') = |V(T')| + 1$ and for the partition $(V_1' = V_1 \cap V(T'), V_2' = V_2 \cap V(T'))$ we have that there exists total dominating sets $S_1'$ and $S_2'$ for $V_1'$ and $V_2'$ both containing $v_4$ such that $\gamma_t(V_1') + \gamma_t(V_2') \leq f_t(T')$. Since $S_1'$ and $S_2'$ contain $v_4$ all end vertices adjacent to $v_4$ must be dominated by $S_1'$ and $S_2'$ and if $S_i := S_i' \cup \{v \in N(v_4) | v \text{ is adjacent to an end vertex } u \in V_i\}$ then $S_i$ is a total dominating set for $V_i$. By observing $S_1$ and $S_2$ we get that $f_t(T) \leq |S_1| + |S_2| \leq n + 1$ and that ii) is satisfied when $d = 6$. □

**Observation 1.**

i) $f_t(G) \leq f_t(G - e)$ if $e \in E(G)$

ii) $f_t(G_1 \cup G_2) = f_t(G_1) + f_t(G_2)$ for graphs $G_1$ and $G_2$.

Now we can find an upper bound on $f_t(G)$ holding for all graphs.

**Theorem 6.**
Let $G$ be a graph with at least four vertices in each component, then $f_t(G) \leq \frac{8}{7}n$ and equality holds if and only if $G$ can be obtained from $\frac{n}{7}P_7$ by adding edges between the center vertices from these $P_7$-paths.

*Proof.* If $G$ is one of the graphs described in the theorem it is easily seen that $f_t(G) = \frac{8}{7}n$. The theorem is shown by induction on $n$ and by Observation 1 we might assume that $G$ is connected. Let $T$ be a spanning subtree of $G$.

If there does not exist an edge $e \in E(T)$ such that the two components of $T - e$ both contain more than four vertices then the theorem follows by Theorem 5. Thus we may assume that there exists an edge $e \in E(G)$ such that the components $T_1$ and $T_2$ of $T - e$ both contain at least four vertices. The induction hypothesis implies that $f_t(T_1) \leq \frac{8}{7}|V(T_1)|$ and $f_t(T_2) \leq \frac{8}{7}|V(T_2)|$. Thus we obtain that $f_t(G) \leq f_t(T) \leq f_t(T_1) + f_t(T_2) \leq \frac{8}{7}n$. If equality holds then $T_1$ and $T_2$ must both be graphs as described in the theorem. Further since this should be the case for any such edge in any spanning subtree it follows that $G$ must have the structure as described in the theorem. □

A set of vertices $S$ in a graph $G$ is said to be a $k$-independent set in $G$ if the distance between any two vertices in $S$ is more than $k$.

**Theorem 7.**
Let $G$ be a connected graph of order $n \geq 3$, then

$$\gamma_t(G) + f_t(k, G) \leq \frac{8}{3}n$$

and equality holds if and only if

1. $G \cong C_3$, $G \cong C_6$ or $G$ is the 2-corona graph for a connected graph.

2. There is a partition $(V_1, \dots, V_k)$ such that each set $V_i$ is a 4-independent set in $G$.

*Proof.* From Theorem 4 we have that $\gamma_t(G) \le \frac{2}{3}n$. Since a set $S$ of vertices is total dominated by each set containing $S$ and a neighbour to each vertex from $S$ we have that $\gamma_t(S) \le 2|S|$. This observation gives $\gamma_t(V_1) + \cdots + \gamma_t(V_k) \le 2|V_1| + \cdots + 2|V_k| = 2n$ for each partition $V_1, \dots, V_k$ of $V(G)$. Assume that $\gamma_t(G) + \gamma_t(V_1) + \cdots + \gamma_t(V_k) = \frac{8}{3}n$ then we must have that $\gamma_t(G) = \frac{2}{3}n$ and $\gamma_t(V_i) = 2|V_i|$. By Theorem 4 we have that $G$ must satisfy (1) and if $V_i$ is not a 4-independent set in $G$ then there exists to vertices $u, v \in V_i$ such that $\gamma_t(\{u, v\}) \le 3$ and we get a contradiction since $2|V_i| = \gamma_t(V_i) \le \gamma_t(\{u, v\}) + \gamma_t(V_i \backslash \{u, v\}) \le 2|V_i \backslash \{u, v\}| + 3 \le 2|V_i| - 1$. If (1) and (2) is satisfied then by Theorem 4 we have that $\gamma_t(G) = \frac{2}{3}n$ and since $V_i$ is a 4-independent set then a total dominating set for $V_i$ must contain two vertices from $N_2[v]$ for each vertex $v \in V_i$. From this observation we have that $\gamma_t(V_i) = 2|V_i|$. $\square$

Let $\mathcal{K}$ denote the graphs that can be obtained from $P_{2k+1} : v_1, v_2, \dots, v_{2k+1}$, for some $k \ge 1$, by attaching a path of length two to each of the vertices $v_1, v_3, \dots, v_{2k+1}$.

**Theorem 8.** Let $T$ be a tree of order $n \ge 4$ that can be obtained by attaching paths of length one or two or stars to vertices from non-adjacent vertices from a path $P$. Then $f_t(T) \le n + 1$ and equality holds if and only if $G \in \mathcal{K}$.

*Proof.* This result is proven by induction on $n$. If $T$ does not have an edge $e$ such that each component of $T - e$ contains at least four vertices then the result is obtained from Theorem 5. If a vertex $v$ is adjacent to more than two leaves and $l$ is one of these leaves then $f_t(T) = f_t(T - l)$ and the result follows by the induction hypothesis.

Let $v_1, \dots, v_k$ be a diametrical path in $T$ and let $i$ be the smallest index such that both components of $T - v_i v_{i+1}$ contains at least four vertices. Let $T_1$ be the component with $v_i$ and $T_2$ be the component with $v_{i+1}$ in $T - v_i v_{i+1}$. By the induction hypothesis we have $f_t(T_1) \le |V(T_1)|$ and $f_t(T_2) \le |V(T_2)| + 1$. Thus we obtain that $f_t(T) \le f_t(T_1) + f_t(T_2) \le n + 1$ and if equality holds then $f_t(T_1) = |V(T_1)|$ and $T_2 \in \mathcal{K}$. If $f_t(T_1) = |V(T_1)|$ and $V(T_1) \ne \{v_1, v_2, v_3, v_4\}$ then for each partition $(V_1', V_2')$ of $V(T_1)$ where $f_t(T_1) = \gamma_t(V_1) + \gamma_t(V_2)$ the vertex $v_i$ is in a $\gamma_t(V_i)$ set for $i \in \{1, 2\}$. Using this observation and considering the graphs from $\mathcal{K}$ it follows that $f_t(T) \le n$ if $V(T_1) \ne \{v_1, v_2, v_3, v_4\}$. If $V(T_1) = \{v_1, v_2, v_3, v_4\}$ and $T_2 \in \mathcal{K}$ it is easy to verify that $T \in \mathcal{K}$ if $f_t(T) = n + 1$. $\square$

**Theorem 9.**
If $n \geq 4$ then $f_t(C_n) \leq n$ and equality holds if and only if $n = 7$ or $n \equiv 0 (\bmod 2)$. If $n \equiv 0 (\bmod 2)$ and $(V_1, V_2)$ is a partition of $C_n$ such that $\gamma_t(V_1) + \gamma_t(V_2) = n$ then for $i \in \{1, 2\}$ and $v \in V(C_n)$ there exists a $\gamma_t(V_i)$-set containing $v$.

*Proof.* The theorem is proven by induction on $n$. It can easily be verified that the theorem holds for $n \leq 12$. Thus we may assume that $n \geq 13$ and the theorem is true for cycles of order less than $n$. By observing $P_5 : v_1, \ldots, v_5$ and a partition $(X_1, X_2)$ we have that $\gamma_t(X_1) + \gamma_t(X_2) \leq 4$ unless $\{v_1, v_5\}$ is a subset of $X_1$ or $X_2$. From this observation, Observation 1 i) and Theorem 8 we see that $f_t(C_n) \leq n$ and vertices from $C_n : v_1, \ldots, v_n$ with distance four between them must have must be in the same set $V_1$ or $V_2$ for a partition of $V(G)$ if $\gamma_t(V_1) + \gamma_t(V_2) = n$.

If $n$ is odd this implies that $\gamma_t(C_n) < n$ and we may assume that $n$ is even. If $n$ is even it can be observed that if the vertices from $C_n$ is alternating then $\gamma_t(V_1) + \gamma_t(V_2) = n$. Further it can be observed that $\gamma_t(V_1) + \gamma_t(V_2) < n$ if there exists a path of length two only with vertices from $V_1$ or from $V_2$. Thus if $\gamma_t(V_1) + \gamma_t(V_2) = n$ and $C_n : u_1, \ldots, u_n, u_1$ we may assume that $V_1 = \{u_i | i \equiv 0 (\bmod 2)\}$ or $V_1 = \{u_i \in V(C) | i \equiv 0 (\bmod 4) \text{ or } i \equiv 1 (\bmod 4)\}$. In both cases it follows that each vertex $v \in V(C_n)$ is contained in a $\gamma_t(V_1)$-set and a $\gamma_t(V_2)$-set. □

Let $\mathcal{T}$ be the set of graphs that can be obtained disjoint copies of $K_{1,3}$ by adding edges between end vertices from these stars such that at most one of the end vertices from a star is incident with one of the added edges.

**Theorem 10.**
If $T$ is a tree of order $n \geq 4$ such that $deg(v) \geq 3$ for each stem $v \in V(T)$ then $f_t(T) \leq n$. Further equality holds if and only if $T \in \mathcal{T}$.

*Proof.* This theorem is proven by induction on $n$. Let $P : u, v, v_1, \ldots, v_k$ denote a diametrical path in $T$. Thus it can easily be assumed that $v$ is adjacent to exactly one leaf $u'$ such that $u' \neq u$. Further we may resume that $T$ has the structure as the graph in figure 1.

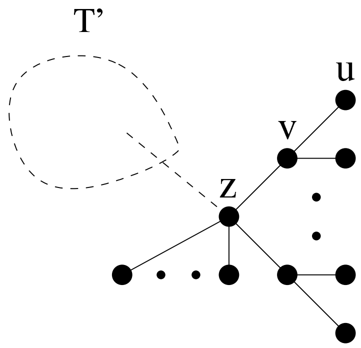


Figure 1: Structure in $T$. $T'$ is not a star with a center at distance two from $v$

If $T'$ is the empty graph then the result easily follows and otherwise we may assume that $|V(T')| \geq 4$. Further it can be observed that a subgraph $G$ of $T$ with $N_T[v] \subseteq V(G)$ has a partition $(V_1', V_2')$ related to $f_t(G)$ such that $v$ and $z$ is contained in minimum total dominating sets for $V_1'$ and $V_2'$. Using this and the induction hypothesis we obtain the result if $N[z]$ contains a stem $s \neq v$.

Let $u, v, v_1, \ldots, v_i$ be a path such that $deg(v_1) = deg(v_2) = \cdots = deg(v_{i-1}) = 2$ and $deg(v_i) \geq 3$. If $v_i$ is adjacent to $z$ then let $T'' := T - zv_i$ and otherwise let $T'' := T - v_{i-2}v_{i-1}$. Thus the induction hypothesis can be applied to the component of $T''$ not containing $v$ and by considering the other component of $T''$ we obtain that $f_t(T) \leq f_t(T'') \leq n$. If equality holds then the component of $T''$ not containing $v$ must be in $\mathcal{T}$ and by verification it can be seen that $T \in \mathcal{T}$. □

Now we examine the graphs $L(n,k)$, defined for integers $n > k \geq 3$, that can be obtained from $C_k \cup P_{n-k}$ by adding an edge between a vertex of degree at most one and a vertex from the circuit. For the graph $L(n,k)$ we define $a := n - k$.

**Theorem 11.**
$f_t(L(n,k)) \leq n$ and if $a \neq 4$ then equality holds if and only if $k = 3$ and $a \in \{1,5\}$, $k = 6$ and $a = 3$ or $k = 7$ and $a \in \{1,5\}$.

*Proof.* Assume that $a = 4$ and let $e$ denote the edge incident with exactly one vertex from the circuit in $L_{n,k}$. It can easily be seen that the theorem holds if $k = 3$ and otherwise it follows by Theorem 8 that $f_t(L_{n,k}) \leq f_t(L_{n,k} - e) \leq n$. Thus it can be assumed that $a \neq 4$. The theorem can easily be verified for the graphs $L(4,3), L(8,3), L(9,6), L(8,7)$ and $L(12,7)$. Let $G = L(n,k)$ denote one of the other graphs from the theorem. Let $P_a : v_1, \ldots, v_a$ denote the path of length $a - 1$ in $G$ such that $deg(v_1) = 1$. By removing paths isomorphic to $P_4$,

$P_5$ and $P_6$ from the end with $v_1$ of the path $P_a$ it follows by Observation 1 that it can be assumed that $a \in \{1, 2, 3\}$.

In the following some cases where $k \geq 7$ is examined. Assume $C_k : u_1, \ldots, u_k$ is a subgraph of $G$ such that $deg(u_4) = 3$. If $a = 1$ and $k \geq 8$ then by observing the graph $G - \{u_1u_2, u_5u_6\}$ or $\{u_1u_2, u_6u_7\}$ it follows by Theorem 8 that $f_t(G) < n$ in this case. If $a \in \{2, 3\}$ and $k \geq 7$ then by observing the graph $G - \{u_2u_3, u_5u_6\}$ or $\{u_1u_2, u_5u_6\}$ it follows by Theorem 8 that $f_t(G) < n$. The rest of the cases can easily be handled by verifications. □

For integers $i \geq j \geq 3$ and $k \geq 1$ we define the graph $D(i, j, k)$ as the graph obtained from $C_i \cup C_j \cup P_k$, where $P_k : v_1, \ldots, v_k$, by adding an edge between $v_1$ and a vertex from $C_i$ and an edge between $v_k$ and a vertex $c_j$ from $C_j$. For $k = 0$ and $i \geq j \geq 3$ the graph $D(i, j, k)$ is defined as the graph obtained from $C_i \cup C_j$ by adding an edge $c_ic_j$ where $c_i \in V(C_i)$ and $c_j \in V(C_j)$.

**Theorem 12.**
Let $G \cong D(i, j, k)$ then $f_t(G) \leq n = i + j + k$ and equality holds if and only if $G \cong D(3, 3, 2)$.

*Proof.* By observing the graph obtained by deleting an edge on $C_i$ incident with $c_i$ it follows by Theorem 11 and Observation 1 that $f_t(G) < n$ unless $i + k = 4$ or $i + k = 5$ and $j = 3$. If $i + k = 4$ or $i + k = 5$ and $j = 3$ then $G \in \{D(3, 3, 1), D(4, 3, 0), D(4, 4, 0), D(3, 3, 2), D(4, 3, 1), D(5, 3, 0)\}$ and the theorem is easily obtained by verification of these graphs. □

Let $\mathcal{G}_1$ be the family of graphs that can be obtained from a cycle $C_{4k}$ by attaching a $P_3$-graph to $2k$ non-adjacent vertices from the cycle. Further let $G_1$ be the graph from $\mathcal{G}_1$ of order 8, $G_2, G_3$ and $G_4$ be the graphs illustrated in figure 2 then the following theorem can be proven.

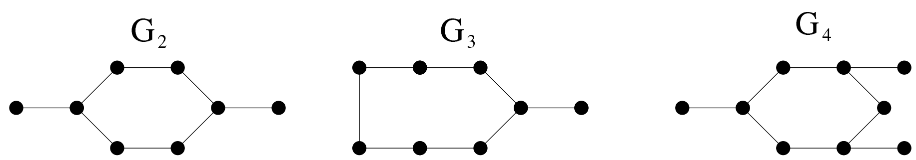


Figure 2: Illustration of the graphs $G_2, G_3$ and $G_4$.

**Theorem 13.**
Let $G$ be a graph of order $n$ that can be obtained from a circuit $C \ncong C_3$ by attaching paths of length one or two or stars to non-adjacent vertices from $C$.

Then $f_t(G) \leq n$ and equality holds if and only if $G \in \mathcal{G}_1 \cup \{G_2, G_3, G_4\}$ or $C \cong C_n$ where $n = 7$ or $n \equiv 0 (\bmod 2)$.

*Proof.* If $G$ is a circuit the theorem follows by Theorem 9. Now let $C : v_1, \ldots, v_k$ where $k \geq 4$ and let $T_{v_i}$ be the component with $v_i$ in the graph obtained from $G$ by deleting the edges incident with $v_i$ on the circuit in $G$. First we assume that there exists a stem $v \in V(G)\backslash V(C)$ adjacent to a vertex from $C$ and $deg(v) \geq 3$. WLOG we may assume that $vv_2 \in E(G)$. By using Theorem 8 on the graph $G - \{v_1v_2, v_3v_4\}$ or $G - \{v_1v_k, v_3v_4\}$ we may assume that $G - V(T_{v_2})$ has order at most four and it easily follows that $f_t(G) < n$ since $G - V(T_{v_2}) \in \{P_3, P_4, K_{1,3}\}$ and $v_2$ is adjacent to exactly two end vertices from $G - V(T_{v_2})$.

Thus it can be assumed that a stem $v$ in $G\backslash V(C)$ satisfy that $deg(v) = 2$ and $v$ is adjacent to a vertex on $C$. Assume that there exists such a stem $v$ in $G\backslash V(C)$ and WLOG it is assumed $v_2$ is adjacent to $v$. By observing the graph $G - \{v_3v_4, v_1v_k\}$ and $G - \{v_4v_5, v_1v_k\}$ if $|V(C_k)| \geq 5$ we may assume that $|V(G - T_{v_2})| \leq 5$ by Theorem 8. By observing the possibilities this gives for $G$ we obtain that $G \in \mathcal{G}_1$ or $f_t(G) < n$.

Thus it can be assumed that vertices from $V(G)\backslash V(C)$ are leaves. Let $v$ be such a leaf and assume WLOG that $vv_3 \in E(G)$. In this case the theorem can easily be verified if $|V(C)| = 4$. Thus it can be assumed that $|V(C)| \geq 5$. If neither $v_1$ nor $v_5$ is adjacent to a leaf it follows by looking at the graph $G - \{v_1v_k, v_4v_5\}$ and $G - \{v_1v_k, v_5v_6\}$ if $k \geq 6$ that $G - T(v_3)$ is of order at most six. By looking at the possibilities this gives for $G$ we obtain that $G \in \{G_2, G_3\}$ when $f_t(G) = n$. Thus it can be assumed WLOG that $v_1$ is adjacent to a leaf. By observing the graph $G - \{v_4v_5, v_1v_k\}$ and $G - \{v_4v_5, v_kv_{k-1}\}$ we may assume that the component of $G - \{v_4v_5, v_1v_k\}$ without $v_3$ is of order at most three. By observing these cases we obtain that $G \cong G_4$ when $f_t(G) = n$. $\square$

**Theorem 14.**
Let $G$ be a connected triangle-free graph of order $n \geq 4$ such that

- $deg(v) \geq 3$ for each stem $v \in V(G)$.
- If $deg(v) \geq 3$, $deg(u) \geq 3$ and $uv \in E(G)$ then $u$ or $v$ is a stem such that all except one of the adjacent vertices are leaves.
- If there exists a path $P : v_1, \ldots, v_k$ such that $k \geq 1$, $deg(v_i) = 2$ for $i \in \{1, \ldots, k\}$ and the neighbours $v$ and $v'$ to $v_1$ and $v_k$ from $V(G)\backslash V(P)$ that either satisfy that $v' = v$ and $deg(v) \geq 4$ or $v' \neq v$ and $deg(v) \geq 3$ then $k \in \{1, 2, 3, 7\}$.

Then $f_t(G) \leq n$.

*Proof.* The theorem is shown by induction on $|E(G)|$. By the induction hypothesis and Observation 1 it can be assumed that $G$ is connected and from Theorem 10 it can be assumed that $G$ contains a circuit $C$. Since $f_t(G) \leq |V(G)|$ if $G \cong C$ (by Theorem 13) it can be assumed that there exists a vertex $v \in V(C)$ such that $deg(v) \geq 3$. If each vertex $x$ adjacent to $v$ is either a leaf, stem adjacent to $deg(x) - 1$ leaves or on $C$ we have by Theorem 13 that the theorem is satisfied (if this is the case for all such vertices like $v$). Thus it can be assumed that there exists a path $P : v = w_0, w_1, \ldots, w_k$ such that $k \geq 1$, $deg(w_i) = 2$ for $i \in \{1, \ldots, k\}$, $w_1 \notin V(C)$ and the neighbour $w_{k+1} \neq w_{k-1}$ to $w_k$ satisfy that $deg(w_{k+1}) \geq 3$. From the assumptions on $G$ it follows that $k \in \{1, 2, 3, 7\}$. If $k = 7$ we have by the induction hypothesis and Observation 1 i) that $f_t(G) \leq f_t(G - \{w_1w_2, w_6w_7\}) \leq n$. If $k = 2$ the bound is obtained from the induction hypothesis and Observation 1 i) by observing the graph $G - w_1w_2$.

If there are no such paths where $k \in \{2, 7\}$ all of the other paths (going from vertices from $C$) must be handled by removing the edge $w_0w_1$ when $k = 1$ and by removing the edge $w_2w_3$ when $k = 3$. Thus we get a graph $G'$ consisting of a component $G''$ isomorphic to one of the graphs from Theorem 13 and the other components satisfy the conditions of this theorem. Thus it follows from Theorem 13, Observation 1 and the induction hypothesis that $f_t(G) \leq f_t(G') \leq f_t(G'') + f_t(G' - V(G'')) \leq |V(G'')| + |V(G') \backslash V(G'')| = |V(G')| = |V(G)|$. Thus the theorem has been proved. $\square$

Finally we can prove an upper bound for $f_t(G)$ when $G$ is a graph with minimum degree at least three.

**Theorem 15.**
Let $G$ be a graph of order $n \geq 4$ such that $\delta(G) \geq 2$, then $f_t(G) \leq n$.

*Proof.* The theorem is proven by induction on the size of $G$. If there exists an edge $uv$ such that $deg(u) \geq 3$ and $deg(v) \geq 3$ then it follows by Observation 1 and the induction hypothesis that $f_t(G) \leq n$ if the components of $G - uv$ contain at least three vertices. Thus it can be assumed in this case that a component of $G - uv$ has order three. If both components are of order three then $f_t(T) = 4 < n$. By Theorem 14 we may assume that $G \ncong C_n$.

Let $P : v_0, v_1, \ldots, v_k$ denote a path in $G$ such that

- $k \geq 1, deg(v_0) \geq 3$
- $deg(v_1) = \cdots = deg(v_k) = 2$
- The neighbour $v_{k+1}$ to $v_k$ from $V(G) \backslash V(P)$ satisfy that $deg(v_{k+1}) \geq 3$ and if $v_{k+1} = v_0$ then $deg(v_{k+1}) \geq 4$.

Assume that $k \notin \{1,2,3,7\}$. If all components in $G' = G - \{v_0v_1, v_kv_{k+1}\}$ have at least four vertices then by Theorem 8 and the induction hypothesis we have that $f_t(G) \leq f_t(G - \{v_0v_1, v_kv_{k+1}\}) \leq n$. If this is not he case $G$ must be a graph with structure as one of the graphs in figure 3 and by using Theorem 8 and the induction hypothesis with the graph $G - \{e, f\}$ we obtain that $f_t(G) \leq n$.

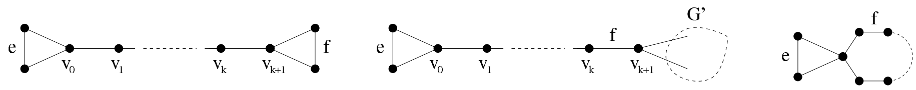


Figure 3: Illustration of structures. In the second graph $G'$ is a connected graph with at least four vertices having at least two vertices from $N(v_{k+1})$.

Thus it can be assumed that $k \in \{1,2,3,7\}$ for such a path. For each subgraph isomorphic to $C_3$ we have that exactly one vertex is adjacent to a vertex from $G - V(C_3)$. If we remove the edges only incident with vertices of degree two from these subgraphs we obtain a graph that satisfy the conditions from Theorem 14. Thus it follows that $f_t(G) \leq n$. □

# 4 Extremal graphs for Theorem 15

It the following section we seek the graphs $G$ with minimum degree at least two that satisfy that $f_t(G) = n$ when all components in $G$ have at least four vertices. From Observation 1 ii) a such graph is extremal if and only if all of its components are extremal. Thus only connected graphs will be examined in the following.

The following theorem gives the extremal graphs for Theorem 14.

**Theorem 16.**
The extremal graphs for Theorem 14 is the graphs $G_2$, $G_3$, $G_4$, $C_n$ where $n = 7$ or $n \equiv 0 (\bmod 2)$ and the trees that can be obtained from the graph $\frac{n}{4}K_{1,3}$ by adding edges between end vertices from these stars such that at most one of the end vertices from a star is incident with one of the added edges.

*Proof.* The theorem is proven by induction on the size of $G$. If $G$ is a tree then the theorem follows by Theorem 10. If $G$ contains only one circuit and the maximum distance to the circuit is two then the theorem follows by Theorem 13. Thus it can be assumed that this is not the case. If $G$ is a extremal graph for Theorem 14 it follows by looking at the proof of Theorem 14 that there must exist two subgraphs $G'$ and $G''$ of $G$ such that these satisfy

1. $f_t(G') = |V(G')|$ and $f_t(G'') = |V(G'')|$
2. One component $H$ of $G'$ is either isomorphic to a graph from $\mathcal{G}_1$ or a extremal graph for Theorem 14 containing a circuit $C$ and the other components of $G'$ is isomorphic to extremal graphs for Theorem 14.
3. $G''$ can be obtained from $G'$ by one of the following two operations:
   (a) Adding an edge $e$ between an end vertex $y$ from $H$ adjacent to a vertex on a cycle and an end vertex $x$ from $G'$ such that the stem $x_s$ adjacent to $x$ satisfy that $deg_{G'}(x_s) \geq 3$. $G''$ is then this graph or the graph obtained by subdividing $e$ five times.
   (b) Adding an edge $e$ incident with an end vertex $x$ from $G'$, where the related stem $x_s$ satisfy that $deg_{G'}(x_s) \geq 3$, and a vertex $y$ from $H$ either on $C$ or at distance two from $C$.

From the induction hypothesis it follows that $H$ is isomorphic to a graph from $\mathcal{G}_1$, $G_2$, $G_3$, $G_4$ or $C_n$ where $n = 7$ or $n \equiv 0 (\bmod 2)$. Further it can be assumed that the girth of $G'$ is the maximum length of a circuit in $G$.

Assume that $G''$ can be obtained from $G'$ by the operation from $3a$. Let $y'$ denote the vertex adjacent to $y$ i $G''$ but not i $G'$. If $y'$ does not belong to $H$ and $G'' - y'$ does not have a component with less that four vertices then 1. and Observation 1 implies that $H + y' + yy'$ must be an extremal graph for Theorem 13 which gives a contradiction since $H \in \{G_2, G_3, G_4\}$ in this case. Otherwise it is easy to obtain a contradiction by some verification.

Thus it can be assumed that $G''$ can be obtained from $G'$ by the operation from $3b$.

First it is assumed that $x \notin V(H)$. Let $X$ denote the component of $G'$ containing $x$. It follows that $f_t(H + x + e) = |V(H)| + 1$ if $|V(X)| > 4$. Thus in the case where $|V(X)| > 4$ we obtain $H \cong C_7$ and by examining $H - y$ and $X + y + e$ it follows that $f_t(X + y + e) \leq |V(X)|$ and $f_t(G'') < |V(G'')|$.

Thus it can be assumed that $|V(X)| = 4$ in this case and $X \cong K_{1,3}$. If $g(H) \neq 4$ or $deg_H(y) = 1$ then by looking at $X+e+y+L_y$, where $L_y$ denotes the leaves adjacent to $y$, and $H-y-L_y$ we obtain a contradiction since $G''$ satisfy 1. If $g(H) = 4$ and $deg_H(y) \neq 1$ then $H+X+e$ can be obtained from a graph from $\mathcal{G}_1 \cup \{C_4\}$ by adding an edge to a star $K_{1,3}$. Since $f_t(H+X+e) < |V(H+X+e)|$ a contradiction is obtained since 1. is not satisfied.

Thus it can be assumed that $x \in V(H)$. In this case it can be assumed that $G' \cong H$. It follows that $H + e$ must be one of the graphs from figure 4.

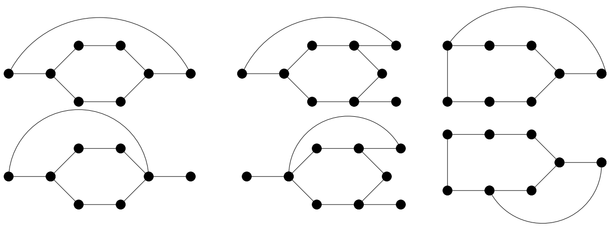

Figure 4: Illustration of graphs.

By observing these graphs we obtain that $H + e$ must be the graph obtained from $G_3$ by adding an edge between the end vertex and a vertex at distance three from it. Since it is assumed that $G$ is a extremal graph for Theorem 14 we must have that $G$ is not isomorphic to $H + e$. By looking at the proof of Theorem 14 it follows that there must be at least one other graph $G'''$ that satisfy the same conditions as $G''$ and where $G'''$ can be obtained from $G''$ by one of the operations from 3).

This gives that the graph obtained from $G_3$ by adding edges from the end vertex to the two vertices at distance three from it must be an extremal graph. Since this is not true the theorem has been proved.

$\square$

Let $\mathcal{F}$ denote the graphs $G$ of order $n \geq 8$ such that $n \equiv 0 \pmod 4$ and $G$ can be obtained from a connected graph $H$, where $|V(H)| = \frac{n}{4}$, by adding a component $C_3$ and an edge between a vertex from this component and $v$ for each vertex $v \in V(H)$.

Further we define $\mathcal{H}$ to be the graphs with components isomorphic to the following graphs :

- The graphs with four vertices with minimum degree at least two.
- The graphs from $\mathcal{F}$ and the graphs from figure 5.
- The graphs $C_n$ where $n = 7$ or $n \equiv 0 \pmod 2$.

The graph from figure 5 that does not have a circuit as a spanning subgraph will be called $G_8$ in the following.

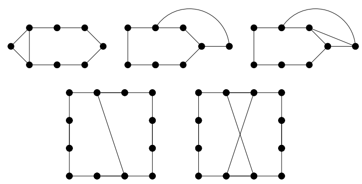

Figure 5: Illustration of the five graphs.

**Lemma 1.**
Let $G$ be a graph from $\mathcal{H}\backslash\{C_7\}$ and let $(V_1, V_2)$ be a partition of $V(G)$ such that $f_t(G) = \gamma_t(V_1) + \gamma_t(V_2)$. If $v \in V(G)$ then there exists a $\gamma_t(V_i)$-set containing $v$ for $i \in \{1, 2\}$.

The following theorem describes the extremal graphs for Theorem 15.

**Theorem 17.**
The extremal graphs for Theorem 15 is $\mathcal{H}$.

*Proof.* Let $G$ be an extremal graph for Theorem 15. The theorem is proven by induction on $|E(G)|$. It easily follows that it can be assumed that $G$ is connected. The theorem easily follows when $n = 4$ since $\gamma_t(V) = 2$ for each non-empty set $V \subseteq V(G)$ in this case.

From Theorem 15 and Theorem 16 it follows that if $G$ is an extremal graph and $G \notin \mathcal{H}$ then $G \in \{D(3, 3, k), D(3, k, 0)\}$ for some integer $k \notin \{1, 2, 3, 7\}$ or there must exist a graph $G'$ in $\mathcal{H}$ such that $G$ can be obtained from $G'$ by one of the following operations :

i) Adding a path $P_k : v_1, \ldots, v_k$ where $k \notin \{1, 2, 3, 7\}$ and edges $v_1y$ and $v_kx$ for vertices $x$ and $y$ from $G'$.

ii) Adding a path $P_k : v_1, \ldots, v_k$ where $k \notin \{1, 2, 3, 7\}$ and a circuit $C_3$ and edges $v_1y$ and $v_kx$ for a vertex $x \in V(C_3)$ and a vertex $y \in V(G')$.

ii) Adding an edge to $G'$.

From Theorem 12 it follows that $G \notin \{D(3, 3, k), D(3, k, 0)\}$ for some integer $k \notin \{1, 2, 3, 7\}$ if $G$ is an extremal graph for Theorem 15 and $G \notin \mathcal{H}$.

Assume that $G$ is obtained from $G'$ by the operation from i). First it is assumed that $G'$ is connected. Assume that $C_{|G'|}$ is a spanning subgraph of $G'$ then $L(n, n-k)$ is a spanning subgraph of $G$ then Theorem 11 implies that $k = 4$

or $k = 5$ and $n \in \{8, 12\}$. By verification it follows that $f_t(G) \leq n$ when $k = 5$ and equality holds if and only if $G \in \mathcal{H}$.

If $k = 4$ the statement is easily obtained by using Theorem 13 , Theorem 11, and Observation 1.

If $G' \cong G_8$ Lemma 1 gives that $f_t(G) \leq f_t(G') + f_t(P_{k-1})$ and it follows that $f_t(G) < n$ unless $k \in \{4, 8\}$. By looking at $G - xv_1$ and $G - yv_k$ it follows that both $x$ and $y$ must have degree two in $G'$ and have two neighbours of degree three. From this observation $G$ can only be four different graphs and by verification the theorem is easily proved in this case. If $G' \in \mathcal{F}$ then it follows that $G$ must have a connected subgraph $G_{xy}$ where $f_t(G_{xy}) \geq |V(G_{xy})|$ and $G_{xy} = 2K_{1,3} \cup P_k + \{v_1x, v_ky\}$ (here $x$ and $y$ belong to different components in $2K_{1,3}$) or $G_{xy} = K_{1,3} \cup P_k + \{v_1x, v_ky\}$ (here $x$ and $y$ belongs to $K_{1,3}$). In these cases it is easy to verify that $f_t(G) < n$ or $G \in \mathcal{H}$.

Here a contradiction is obtained since it follows from Theorem 13 or Theorem 10 that $f_t(G_{xy}) < |V(G_{xy})|$.

Assume $G'$ is not connected and let $G_v$ denote the component containing $v$ for $v \in \{x, y\}$. If both $G_x$ and $G_y$ have circuits as spanning subgraphs it follows from Theorem 12 that $f_t(G) < n$. If $G_x$ has a circuit as a spanning subgraph then by observing $G - v_ky$ and using Theorem 11 it follows that $k = 4$ or $|V(G_x)| = 7$ and $k = 5$. Since it can be assumed that $G_y \ncong C_7$ in this case it follows from Lemma 1 and Theorem 11 that

$$f_t(G) \leq f_t(G - v_{k-1}v_k) = f_t(G - v_{k-1}v_k) + f_t(L(|G_x|, k-1) + f_t(G_y) < n.$$

Thus it can be assumed that $G_x$ does not have a circuit as a spanning subgraph and by the same arguments the same must hold for $G_y$. Thus it can be assumed that both $G_x$ and $G_y$ are contained in $\mathcal{F} \cup G_8$. Since Lemma 1 gives that $f_t(G) \leq f_t(G - \{v_1v_2, v_{k-1}v_k\}) = f_t(G_x) + f_t(G_y) + f_t(P_{k-2})$ it follows that $k = 4$ since $f_t(G) = n$. The theorem now easily follows by verification.

Assume that $G$ is obtained from $G'$ by the operation from ii). If $G' \ncong C_7$ we easily obtain $f_t(G) < n$ by using Lemma 1 and Theorem 11. Thus it can be assumed that $G \cong D(7, 3, k)$ and since $f_t(G) < n$ in this case a contradiction is obtained.

Thus it can be assumed that $G$ is obtained from $G'$ by the operation from iii). Let $xy$ denote the edge added to $G'$ and let $G_v$ denote the component of $G'$ containing $v$ for $v \in \{x, y\}$. First it is assumed that $G'$ is not connected. If $G_x \ncong C_7$ it follows from Lemma 1 that $f_t(G) \leq f_t(G_x) + f_t(G_y - y)$ and further it must hold that $f_t(G_y - y) = f_t(G_y)$. From this observation it follows that $C_8$ is a spanning subgraph of $G_y$ or $G_y - y$ has a component with at most three vertices. By using Lemma 1 we obtain that $C_8$ is a spanning subgraph of $G_x$ or $G_x - x$ has a component with at most three vertices. By verifications it is easy to obtain a contradiction with $f_t(G) = n$. Similar a contradiction is obtained if $G_y \ncong C_7$. Thus it can be assumed that $G \cong D(7, 7, 0)$ and since $f_t(G) < |V(G)|$ in this case a contradiction is obtained. Thus it can be assumed that $G'$ is connected. If $G'$ has $C_n$ as a spanning subgraph and $e$ is an edge on $C_n$ adjacent to $x$ then it must hold that $f_t(G) = n = f_t(C_n + xy - e)$. From this observation and Theorem 11 it can be concluded that $n \in \{4, 8, 10, 12\}$ and further it must hold that $d_{C_n}(x, y) \in \{2, 4\}$ when $n = 8$, $d_{C_n}(x, y) = 4$ when $n = 10$, and $d_{C_n}(x, y) = 6$ when $n = 12$. By verification it follows that $G \in \mathcal{H}$ when $f_t(G) = n$. Further the theorem easily follows by verification when $G'$ has the graph $L(8, 7)$ as a spanning subgraph. Thus it can be assumed that $G' \in \mathcal{F}$. If $G \notin \mathcal{H}$ then it follows by using Observation 1 that the graph $G''$ obtained by adding an edge between vertices not both of degree one in $2 * L(4, 3)$ must be a subgraph of $G$ that satisfies $f_t(G'') = |V(G'')|$. Since this gives a contradiction the theorem has been proved in this case. $\square$

## References


[1] T.W. Haynes, S.T. Hedetniemi, and P.J. Slater, *Fundamentals of Domination in Graphs.* Marcel Dekker, New York, 1998.

[2] T.W. Haynes, S.T. Hedetniemi, and P.J. Slater (eds.), *Domination in Graphs: Advanced Topics.* Marcel Dekker, New York, 1998.

[3] E.J. Cockayne, R.M. Dawes, and S.T. Hedetniemi *Total domination in graphs. Networks* **10**, 211–219.

[4] S. Seager *Partition dominations of graphs of minimum degree 2. Congr. Numer* **132** (1998) 85–91

[5] M. R. Garey *Computers and Intractability: A Guide to the Theory of NP-Completness. Freeman, New York* (1979)